Received: *November 22, 1999*, Accepted: *August 13, 2000*

HYPER VERSION

D-branes on the quintic

Ilka Brunner[a], **Michael R. Douglas**[a,b]

[a] *Department of Physics and Astronomy, Rutgers University*
Piscataway, NJ 08855–0849, USA
[b] *I.H.E.S.*
Le Bois-Marie, Bures-sur-Yvette, 91440 France
E-mail: ibrunner@physics.rutgers.edu, mrd@pion.Rutgers.EDU

Albion Lawrence

Department of Physics, Harvard University
Cambridge, MA 02138, USA
E-mail: albion@yukawa.uchicago.edu

Christian Römelsberger

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855–0849, USA
E-mail: roemel@physics.rutgers.edu

Abstract: We study D-branes on the quintic CY by combining results from several directions: general results on holomorphic curves and vector bundles, stringy geometry and mirror symmetry, and the boundary states in Gepner models recently constructed by Recknagel and Schomerus, to begin sketching a picture of D-branes in the stringy regime. We also make first steps towards computing superpotentials on the D-brane world-volumes.

Keywords: Topological Field Theories, Solitons Monopoles and Instantons, D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction

In this work we study D-branes on the quintic Calabi-Yau, historically the first CY to be intensively studied. Our guiding question will be: to classify all supersymmetry-preserving D-branes at each point in CY moduli space, and find their world-volume moduli spaces. As is well known, results of this type are quite relevant for phenomenological applications of M/string theory, because the world-volume theories we will obtain include a wide variety of four-dimensional theories with $\mathcal{N}=1$ space-time supersymmetry. The problem includes the classification of holomorphic vector bundles (which are ground states for wrapped six-branes); and almost all M/string compactifications which lead to $d=4$, $\mathcal{N}=1$ supersymmetry (such as $(0,2)$ heterotic string compactifications and F theory constructions) have a choice of bundle as one of the inputs. Thus, many works have addressed this subject explicitly or implicitly.

As usual in string compactification this geometric data is only an input and one would really like to answer the same questions with stringy corrections included. The primary question along these lines is: is the effect of stringy corrections just quantitative — affecting masses and couplings in the effective lagrangian but preserving the spectrum and moduli spaces — or is it qualitative? If the latter, we might imagine that geometric branes undergo radical changes of their moduli space or are even destabilized in the stringy regime, with new branes which were unstable in the large volume limit taking their place. It should be realized that at present very little is known about this question; for example it has not been ruled out that the D0-brane becomes unstable in the stringy regime or has moduli space dimension different from 3.

Clearly these questions are of great importance for the string phenomenology mentioned above and were asked long ago in the context of $(0,2)$ models. No simple answer has been proposed; we will return to this in the conclusions.

A concrete framework which allows an exact CFT study of the stringy regime is provided by the Gepner models. The main lesson from the original study of Gepner models for type-II and heterotic strings was that these CFT compactifications are continuously connected to CY compactification. Mirror symmetry is manifest in the 2d superconformal field theory, and this connection was one of the earliest arguments for it in the CY context.

The first detailed study of D-branes in Gepner models was made by Recknagel and Schomerus [7] who (following the general approach of Cardy) constructed a large set of examples; further work appears in [8, 9]. So far no geometric interpretation or contact with the large volume limit has been made. We will do so in this work. The main tool we will use is the (symplectic) intersection form for three-cycles in the large volume limit. This form governs Dirac quantization in the effective $d = 4$ theory and as such must be invariant under any variation of the moduli. As argued in [10, 11] it is given by the index $\mathrm{Tr}_{ab}(-1)^F$ in open string CFT and thus is easily computed for the Gepner boundary states. The detailed study of Kähler moduli space by Candelas et. al. [12] then allows relating this to the large volume basis for $2p$-branes. We can also make the large volume identification for the 3-branes, aided by the large discrete symmetry group.

The detailed outline of the paper is as follows. In section 2 we review the quintic, its homology and moduli space, and give a general overview of D-branes on the quintic in the large volume limit. In section 3 we review the stringy geometry of its Kähler moduli space and the monodromy group acting on B branes. In section 4 we review Gepner models and Cardy's theory of boundary states, which will allow us to review the boundary states constructed by Recknagel and Schomerus. We briefly discuss the theory for K3 compactifications, and show that the results agree with geometric expectations; in particular that the dimension of a brane moduli space on K3 is given by the Mukai formula. In section 5 we compute the large volume charges for the quintic boundary states, and compute the number of marginal operators. This will allow us to propose candidate geometric identifications. In section 6 we discuss the computation of world-volume superpotentials. We begin by presenting evidence that the superpotential is "topological" in a sense that we explain. If true, an important consequence would be that the superpotential for B-type branes has relatively trivial Kähler dependence and can thus be computed in the large volume limit. This would imply general agreement between stringy and geometric results, analogous to the case of the prepotential. In section 7 we discuss superpotential computations in the Gepner model and derive selection rules. Besides charge conservation rules similar to those in the closed string sector, additional boundary selection rules appear, and we illustrate these with the examples of the A_1 and A_2 minimal models. The selection rules will allow us to establish that certain branes have non-trivial moduli spaces. The exact superpotentials should be calculable given the solutions of the consistency conditions of boundary CFT [13, 14]; this is work in progress. In section 8 we summarize our results and draw conclusions.

A point of notation: in labeling a p-brane, we always ignore its Minkowski space-filling dimensions (for example, a D4 wraps four dimensions of the CY), but we describe its world-volume lagrangian in $d = 4$, $N = 1$ terms (appropriate if the brane filled all $3 + 1$ Minkowski dimensions).

JHEP08(2000)015

2. Large volume limit of the quintic

2.1 General discussion of D-branes on large volume CY

We are interested in BPS states in type-II string theory described by collections of D-branes at points on or wrapping some cycle in a Calabi-Yau manifold M. A configuration for N coincident D-branes with worldvolume Σ wrapped on such a cycle is specified by an embedding $X : \Sigma \to M$ and a $\mathrm{U}(N)$ gauge field A on Σ, with field strength $F = dA + [A, A]$. The $\mathrm{U}(1)$ part of $\mathrm{U}(N)$ appears in combination with the B-field, $\mathcal{F} = F - X^*B$, where X^*B is the pullback of the NS B-field onto the worldvolume.

The conditions for supersymmetric embeddings with nonabelian fields turned on has not been given, but they have been worked out for single D-branes in refs. [3, 15], for which the action of spacetime supersymmetry and worldvolume κ-symmetry is known [16]. A compactification preserving supersymmetry will occur if there are constant spinors η^i on M for each of the spacetime SUSYs. These supersymmetries transform the embedding coordinates (and their superpartners) on the D-brane worldvolumes; they are preserved if one can find a κ-symmetry transformation which cancels the SUSY transformation. This condition can be written as

$$(1 - \Gamma)\eta^i = 0 \tag{2.1}$$

and those η^i which are solutions form the unbroken SUSYs. Γ is defined as follows [15]. Let E^m_μ be the vielbein connecting frame indices m and spacetime indices μ. We can pull this back to the worldvolume, defining

$$E^m_\alpha = \partial_\alpha X^\mu E^m_\mu(X)\,, \tag{2.2}$$

where α is a worldvolume index for the p-brane. With this we can pull back the 10D γ-matrices Γ_m:

$$\Gamma_\alpha = E^m_\alpha \Gamma_m\,. \tag{2.3}$$

Define

$$\Gamma_{(p+1)} = \frac{1}{(p+1)!\sqrt{g}}\epsilon^{\alpha_1\ldots\alpha_{p+1}}\Gamma_{\alpha_1\ldots\alpha_{p+1}}\,, \tag{2.4}$$

where

$$g_{\alpha\beta} = \eta_{mn}E^m_\alpha E^n_\beta \tag{2.5}$$

is the induced metric on the Dp-brane. We can now write:

$$\Gamma = \frac{\sqrt{g}}{\sqrt{g+\mathcal{F}}}\sum_{\ell=0}^{\infty}\frac{1}{2^\ell \ell!}\Gamma^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\mathcal{F}_{\alpha_1\beta_1}\ldots\mathcal{F}_{\alpha_n\beta_n}\Gamma^{n+(p-2)/2}_{(11)}\Gamma_{(p+1)}\,. \tag{2.6}$$

When $\mathcal{F} = 0$ this can be written in the simpler, more familiar form:

$$\Gamma = \epsilon^{\alpha_1\ldots\alpha_{p+1}}\partial_{\alpha_1}X^{\mu_1}\ldots\partial_{\alpha_{p+1}}X^{\mu_{p+1}}\Gamma_{\mu_1}\ldots\Gamma_{\mu_{p+1}}\,, \tag{2.7}$$

JHEP08(2000)015

where $\Gamma_\mu = E_\mu^m \Gamma_m$. The conditions in this latter case have been worked out in some detail, as we will describe below. These conditions match those in refs. [17, 18] for boundary states of BPS D-branes in flat space with constant background fields.

Solutions to eq. (2.1) in the presence of nonzero $\mathcal{F}$ have been worked out for flat, intersecting branes in refs. [17, 18, 15]. In the case of BPS D-branes in Calabi-Yau 3-fold compactifications the geometric conditions implied by (2.1) (and the analog for boundary states) have been worked out in [3, 6]. These solutions fall into two classes: "A-type" branes wrapping special lagrangian submanifolds and "B-type" branes wrapping holomorphic cycles. Let us describe each of these in turn.

2.1.1 B branes

"B-type" BPS branes wrap even-dimensional, holomorphic cycles in the Calabi-Yau [3, 6]. For B (even-dimensional) branes, (2.1) is solved by holomorphically embedded curves (2-branes) and surfaces (4-branes), as well as by 0 and 6-branes with the obvious (trivial) embeddings. We may also have gauge fields on these branes. In general the gauge field may change the definition of a supersymmetric cycle via eq. (2.1). However, if the brane is wrapped around a holomorphic cycle, we can find conditions for the gauge field to preserve the supersymmetries. In the case of N coincident D6-branes wrapping the entire CY threefold, if we assume that the gauge fields live only in the threefold then the SUSY-preserving gauge field must satisfy the "hermitean Yang-Mills equations" [19]:

$$F_{ij} = 0 \omega^2 \wedge \operatorname{tr} F = c\omega^3 \,, \tag{2.8}$$

where (i, j) and $\bar{\imath}, \bar{\jmath}$ are holomorphic and antiholomorphic indices, respectively, on the CY. These equations define a "hermitean-Einstein" connection A with curvature F. The first equation tells us that the vector bundle is holomorphic. The second equation tells us that the vector bundle is "ω-stable"; conversely, ω-stability guarantees a solution to these equations [20] (c.f. chapter 4 of [21] for a discussion and definitions.)

For branes wrapped around holomorphic submanifolds of M, these equations must be altered. The gauge fields polarized transverse to the cycle are replaced by "twisted" scalars Φ which are one-forms in the normal bundle to the embedding [4], and eq. (2.8) becomes a generalization of the Hitchin equations for Φ and F [19].

It is believed that all topological invariants of a D-brane configuration are given by an element of a particular K-theory group on M [22, 23]. When the K-theory group and/or the cohomology of M has torsion the K-theory interpretation is important; one may have objects charged under the torsion. The charge can be written [22] as a generalization of the results of [24, 25]:

$$v(E) = \operatorname{ch}(f_! E)\sqrt{\hat{A}(M)} \,. \tag{2.9}$$

Here E is a vector bundle on Σ; remember that we must extend the U(1) part of the gauge field F by the NS B-field, so properly the vector bundle E is a polynomial in $\mathcal{F}$.

JHEP08(2000)015

Let $\pi : M \to \Sigma$ be the projection onto the worldvolume and N be the normal bundle of $\Sigma \hookrightarrow M$. There is a K-theory element $\delta(N)$ which is roughly a delta function on the worldvolume and depends on N; we can thus define $f_!(E) = \pi^* E \otimes \delta(N)$. The moduli space of D-branes will not just be the moduli space of vector bundles in this K-theory class but rather the moduli space of coherent semistable sheaves in this class [26, 19]. Some advantages of this definition through K-theory and sheaves, besides the fact that it seems to be correct, are that it places configurations with D6-branes (gauge field configurations on M) on an equal footing with configurations without D6-branes, and that it can describe certain singularities which lead to sensible string compactifications.

In examples without torsion, such as the quintic, one may describe the D-brane charge in a less esoteric fashion. Assuming the branes give rise to particles in the macroscopic directions, for a $2n$-dimensional worldvolume Σ we can write the D-brane coupling to the RR gauge fields via the "Wess-Zumino term" as [27, 24, 25]:

$$\int_\Sigma C \wedge \mathrm{ch}(F - B) \sqrt{\frac{\hat{A}(M)}{\hat{A}(N)}}\,, \tag{2.10}$$

where

$$C = C^{(2n+1)} + C^{(2n-1)} + \cdots + C^{(1)}$$

is a sum over the (k)-form RR potentials that couple to the $2n$-brane.

These RR charges reduce to conventional electric and magnetic charges in the four noncompact dimensions. Given two D-branes which reduce to particles, the most basic observable we can study is the Dirac-Schwinger-Zwanziger symplectic inner product on their charges,

$$I(a, b) = Q_{Ea} \cdot Q_{Mb} - Q_{Ma} \cdot Q_{Eb}\,. \tag{2.11}$$

We will refer to this as the "intersection form" as it is closely related to the topological intersection form for two- and four-branes. For two six-branes, from the formulas above it is

$$I(a, b) = \int \mathrm{ch}(F_a)\ \mathrm{ch}(-F_b)\ \hat{A}(M)\,. \tag{2.12}$$

Finally, we quote a general theorem regarding stability (Bogomolov's inequality [28]; c.f. [29, 21]): given a variety X of dimension n and ω an ample divisor on X, then a ω-semistable torsion free sheaf E of rank r and Chern classes c_i will satisfy

$$\int_S \left(2r\ c_2 - (r-1)c_1^2\right) \wedge \omega^{n-2} \geq 0\,. \tag{2.13}$$

The parenthesized combination is called the "discriminant" of the sheaf and is equal to $c_2(\mathrm{End}(E))$. In the special case $c_1(E) = 0$ this amounts to requiring $c_2(E) \geq 0$.

JHEP08(2000)015

2.1.2 A branes

An "A-type" BPS brane wraps a three-dimensional special lagrangian submanifold Σ [3]:[1]

$$\begin{aligned} \omega|_\Sigma &= 0 \\ \mathrm{Re}\, e^{i\theta}\Omega|_\Sigma &= 0\,. \end{aligned} \tag{2.14}$$

Here Ω is the holomorphic 3-form of the Calabi-Yau and θ is an arbitrary phase. Equivalently to the second equation, we can require that Ω pulls back to a constant multiple of the volume element on Σ. Furthermore the gauge field on this manifold must be flat. A nice introduction to the general theory of these is [31]. It is shown there (and in the references therein) that the moduli space has complex dimension $b^1(\Sigma)$. The space of flat U(1) connections has real dimension $b^1(\Sigma)$, and ω^{ij} can be used to get an isomorphism between $T^*\Sigma$ and $N\Sigma$; thus the deformations of Σ pair up with the Wilson lines to form $b^1(\Sigma)$ complex moduli.

For three-branes, the DSZ inner product (2.11) is precisely the geometric intersection form.

One application of these branes is the Strominger-Yau-Zaslow formulation of mirror symmetry, a precise formulation of the idea that "mirror symmetry is T-duality" [32]. Since mirror symmetry exchanges the sets of A and B branes, an appropriately chosen moduli space of A branes on M will be the moduli space of D0-branes on the mirror W. Clearly $b^1 = 3$ for such A-branes, and SYZ argue that Σ will be a T^3 in this case. A similar proposal was made for general B branes with bundles in [33].

Another application is the construction of $N = 1$ gauge theories with the help of brane configurations. Supersymmetric three-cycles have been used to explore the strong coupling limit by lifting the brane configurations to M-theory in [34].

Not too many explicit constructions of special lagrangian submanifolds are known and it appears (e.g. see [31]) that the problem is of the same order of difficulty as writing explicit Ricci-flat metrics on a CY. A general construction we will use below is as the fixed point set of a real involution.

2.2 General world-volume considerations

Given a system X of A or B D-branes, we can consider the system which is identical except that it extends in the flat $3+1$ dimensions transverse to M. This system will have a $d = 4$, $\mathcal{N} = 1$ supersymmetric gauge theory as its low-energy world-volume theory, whose data is a gauge group G_X; a complex manifold C_X parameterized by chiral superfields ϕ^i; a Kähler potential K on C_X; an action by holomorphic isometries of G_X on C_X (linearizing around a solution this corresponds to the usual choice of representation R of the gauge group), and a superpotential W (a holomorphic

[1] There is some evidence that the special lagrangian condition receives α' corrections [30].

function on C_X invariant under the action of G_X). If G_X contains U(1) factors, each of these can have an associated real constant ζ_a (the "Fayet-Iliopoulos terms").

In the classical ($g_s \to 0$) limit, the moduli space of this theory is the solutions of $F_i = \partial W/\partial\phi^i = 0$ (the "F-terms") and $D^a = \zeta^a$ (the "D-terms") modulo gauge transformations, where D^a is the moment map generating the associated gauge transformation (and $\zeta^a \equiv 0$ in the non-abelian parts of the gauge group).

We review this well-known material for a number of reasons. First, we remind the reader that although some of our later discussion will use other realizations of this D-brane system (for example as particles in $3+1$ dimensions), the world-volume theories for these other realizations are all obtained by naive dimensional reduction from the $3+1$ theory (if $g_s \sim 0$), while the $3+1$ language makes it easy to impose supersymmetry.

Second, it is known that the study of bundles and sheaves on CY three-folds is much more complicated than that for K3; this complication has a direct physical counterpart in the reduced constraints of $\mathcal{N} = 1$ supersymmetry. The most basic example of this is the fact that — unlike the case for K3 — there is no formula for the dimension of the moduli space of E given $c(E)$. The main reason for this is that this dimension is not necessarily constant — the moduli space can have branches of different dimension, and can depend on the moduli of the CY as well.

Physically, this corresponds to the possibility of a fairly arbitrary superpotential in the low energy theory. Indeed, the language of superpotentials and $\mathcal{N} = 1$ effective lagrangians might be the best one for these problems, much as hyperkahler geometry and hyperkahler quotient is for instanton problems in four dimensions. Just as the self-dual Yang-Mills equations can be regarded as an infinite-dimensional hyperkahler quotient, we might pose the problem of rephrasing the YM equations under discussion as the problem of finding the moduli space of an $\mathcal{N} = 1$ effective theory with an infinite number of fields.

The basic outlines of part of this treatment are known (see [35, chapter 6], for a very clear discussion of the four-dimensional case). The two equations (2.8) will correspond directly to the F-term (superpotential) constraints and the D-term constraints, respectively. Indeed, the problem of solving $F_{ij} = 0$ is a purely holomorphic problem, while it is not hard to see that the expression $F^a \wedge \omega^{n-1}$ is the moment map generating conventional gauge transformations. The stability condition on the bundle is exactly the infinite-dimensional counterpart of the usual condition in supersymmetric gauge theory for an orbit of the complexified gauge group to contain a solution of the D-flatness conditions (e.g. see [36]). Donaldson's theorem proving the existence of such solutions proceeds exactly by considering the flow generated by i times the moment map to a minimum; the Uhlenbeck-Yau generalization is quite similar (for technical reasons a different equation is used).

The other part of the story — translating the problem of finding holomorphic vector bundles into solving constraints on a finite-dimensional configuration space, which can be derived from a superpotential — does not seem to have been addressed in as systematic a manner; clearly this could be useful.

In a sense the six-dimensional problem is the "universal" one which also describes the lower-dimensional branes. Not only can their charges be reproduced, but gauge field singularities will correspond to specific lower dimensional branes (e.g. the small instanton). Furthermore, there is a sense in which even the lower-dimensional brane world-volume theories are six-dimensional if we include "winding strings" (by analogy to tori and orbifolds, although this idea has not yet been made precise). Treating a system of N D0's as quantum mechanics requires neglecting these strings, which one expects to be problematic once the separation between branes approaches the size of the space.

We now turn from these abstract ideas to our concrete example.

2.3 D-branes on the quintic

Perhaps the best-studied family of Calabi-Yau manifolds is the quintic hypersurfaces in $\mathbb{P}^4$. A relatively thorough discussion of these is contained in the classic paper [12]. The moduli space of these manifolds is locally the product of $b_{2,1} = 101$ complex structure deformations and $b_{1,1} = 1$ deformations of the complexified Kähler forms $B + iJ$ (where B is the flux of the NS-NS B-field). We will be particularly interested in the Fermat quintic

$$P = \sum_{i=1}^{5} z_i^5 = 0\,, \tag{2.15}$$

where z_i are the homogeneous coordinates on $\mathbb{P}^4$. Note that this equation has a $S_5 \bowtie \mathbb{Z}_5^4$ discrete symmetry; the Z^5 generators are $g_i : z_i \to \omega z_i$ and satisfy the relation $\prod_{i=1}^{5} g_i = 1$, while the S_5 permutes the coordinates in the obvious way.

2.3.1 B branes on the quintic

As we have discussed, D-branes on the quintic can be described by vector bundles or sheaves on this space. Let us denote the charge carried by a single D$2p$-brane wrapped about a generator of H^{2p} as $Q_{2p} = 1$.

Transporting a D-brane configuration about closed, nontrivial cycles of the moduli space of Kähler structures will induce an associated $\mathrm{Sp}(4, \mathbb{Z})$ monodromy on the B branes. We will discuss the monodromy more completely in the next section, but there is already one cycle in the moduli space which can be understood in the large volume limit: $B \to B + 1$, where B is the NS 2-form. The action on the charge Q can be seen from eqs. (2.9), (2.10) [37]. Mathematically this corresponds to the possibility to tensor the vector bundle V_{2p} with a U(1) bundle of $c_1 = 1$. This preserves stability and the dimension of the moduli space. Given a bundle V this operation and its inverse can be used to produce a related bundle with $-r < c_1 \leq 0$: this is referred to as a "normalized" vector bundle.

There is no classification of vector bundles and coherent sheaves on the quintic, but we can write down a few examples in order to orient ourselves when discussing specific boundary states at the Gepner point.

BPS D2-branes wrap holomorphic 2-cycles of the Calabi-Yau, the same cycles as appear in worldsheet instanton corrections. Such cycles can have arbitrary genus and arbitrary degree. Degree one rational curves are generically rigid on the quintic [38]. Nontheless for special quintics, families may exist; for example, in the case of the Fermat quintic (2.15), there are 50 one-parameter families essentially identical to the family [39, 40]:

$$\begin{aligned}&(z_1, z_2, z_3, z_4, z_5) = (u, -u, av, bv, cv)\\ &a^5 + b^5 + c^5 = 0\,; \qquad a, b, c \in \mathbb{C}\,,\end{aligned} \tag{2.16}$$

where (u, v) are homogeneous coordinates in $\mathbb{P}^1$. Once we perturb away from the Fermat point, these moduli are lifted and a finite number of rational curves remain [39]. This could be described in the world-volume theory by a superpotential of the general form

$$W = \phi\psi^2\,,$$

where ϕ are complex structure moduli; $\phi = 0$ is the Fermat point; ψ are curve moduli, and $\psi = 0$ a curve which exists for generic quintics.

The infinitesimal description of deformations of such cycles is as sections of the normal bundle, which by the Calabi-Yau condition will be $\mathcal{O}(a)\oplus\mathcal{O}(b)$ with $a+b = -2$ for a rational curve. One might think that all one needs to find examples of families is to find examples with $a \geq 0$ or $b \geq 0$, but this is not true as deformations can be obstructed. The canonical example is given by resolving the singularity in $\mathbb{C}^4$

$$xy = z^2 - t^{2n}\,. \tag{2.17}$$

For $n = 1$ this is the conifold singularity and the "small" resolution contains a rigid $\mathbb{P}^1$, parameterized by $x/(z - t) = (z + t)/y$. It can be shown [41] that for $n > 1$ the resolution also contains a $\mathbb{P}^1$, now with normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$, but the deformation is obstructed at n'th order, as could be described by the superpotential

$$W = \psi^{n+1}\,. \tag{2.18}$$

Intuitively this can be seen by deforming (2.17) by a generic polynomial in t^2, which splits the singularity into n conifold singularities, each admitting a rigid $\mathbb{P}^1$. If we then tune the parameters to make these $\mathbb{P}^1$'s coincide, a superpotential describing the n vacua will degenerate to (2.18). Such singularities do appear in large families of quintic CY's [38].[2]

[2](Note added in v2): The idea that the moduli space of such a curve can always be described as the critical points $W' = 0$ of a single holomorphic function was apparently not known to mathematicians. We thank S. Katz for a discussion on this point.

It turns out that the curves in (2.16) provide another example of obstructed deformations [39].[3] The normal bundle of these curves is $\mathcal{N} = \mathcal{O}(1) \oplus \mathcal{O}(-3)$; as $\dim H^0(\mathcal{N}) = 2$, there must be another obstructed deformation; call it ρ. The correct counting of curves upon deforming away from the Fermat point can be reproduced by a superpotential ρ^3. The modulus ρ is also connected to the fact that pairs of the 50 families in (2.16) intersect (e.g. take (2.16) and the family $(av, bv, u, -u, cv)$ with $c = 0$); it describes deformations into the second family. All of this structure can be summarized in the superpotential

$$W(\rho, \psi) = \rho^3\psi^3 + \phi F(\rho, \psi) + \cdots ;$$

where ϕF generalizes the $\phi\psi^2$ term discussed above.

Higher genus curves can generically come in families and examples can be found as complete intersections of hypersurfaces in $\mathbb{P}^4$ with the quintic. A particular example is the intersection of two hyperplanes with the quintic [40]:

$$\sum_{k=1}^{5} a_k z_k = \sum_{k=1}^{5} b_k z_k = 0\,, \qquad a_k, b_k \in \mathbb{C}\,. \tag{2.19}$$

It is easy to see that there are six independent complex parameters after rescaling the equations. The curve is genus 6, and the area of the curve C is $\int_C J = 5$, where J is the unit normalized Kähler form of $\mathbb{P}^4$, i.e.

$$\begin{aligned} \int_{\mathbb{P}^4} J \wedge J \wedge J \wedge J &= 1 \\ \int_{\text{Quintic}} J \wedge J \wedge J &= 5 \end{aligned} \tag{2.20}$$

Thus this brane has $Q_2 = 5$.[4] There will be six additional complex moduli coming from Wilson lines of the U(1) gauge field around the 12 cycles of the curve.

Similarly, four-branes can be obtained as the intersection with another hypersurface in $\mathbb{P}^4$. For example, the intersection of the quintic with a single hyperplane

$$\sum_k a_k z_k = 0$$

produces a four-parameter family of four-cycles S. Their volume is $\int_S J = 5$ and so $Q_4 = 5$. In addition $c_2(TS) = 11J^2$; so that the coupling of $C^{(1)}$ to $p_1/48$ in eqs. (2.9), (2.10) leads to an induced 0-brane charge of 55/24. The four-brane generically may support nontrivial gauge field flux over two-cycles, corresponding to D2-brane charge,

[3](Note added in v3): We would like to thank S. Katz for explaining this example, pointing out a mistake in our earlier draft, and suggesting the superpotential discussed here.

[4]See [42, chapters 1 and 2], for a nice description of complete intersections in projective spaces, and of techniques for performing the calculations we allude to here.

or instanton solutions, corresponding to zero-brane charge. Some discussion of the moduli space of four-branes in a Calabi-Yau can be found in [43]. By (2.13), stability of the vector bundle on the four-brane requires $Q_0 > 0$.

Finally we can look at the case of D6-branes wrapping the entire Calabi-Yau manifold. In fact we will find that all of the boundary states we examine at the Gepner point will have non-trivial six-brane charge. A single six-brane by itself will have no moduli. The U(1) gauge field on a single 6-brane can support flux with first Chern class $c_1 = n$ corresponding to $Q_4 = n$. We can get the relevant bundles by restriction from U(1) bundles on $\mathbb{P}^4$. The latter have no moduli, and we will not gain any upon restriction.

We can also imagine binding D2-branes to the D6-brane, by analogy to $2-6$ (or $0-4$) configurations in flat space. For $Q_6 = 1$ and $Q_4 = 0$ this appears singular; U(1) gauge fields do not support smooth instanton solutions. The brane counterpart to this is that the $2-6$ strings cannot be given vevs which bind the branes and give mass to the relative U(1)s. This might lead us to predict that such states, if they exist at all, exist only as quantum-mechanical bound states. Such a state should be easily identifiable because it appears at the junction of Coulomb and Higgs branches of the moduli space; a small perturbation should put it on the Coulomb branch and produce two U(1) gauge fields in the macroscopic direction. In the classical considerations of this paper, it should not show up at all.

For $Q_6 > 1$, we require information about vector bundles on the Calabi-Yau. A well-known example with $Q_6 = 3$ is deformations of the tangent bundle. This has vanishing c_1 and $c_2(E) = 10J$ giving us $Q_2 = 50$. The dimension of the moduli space is 224. This example can be generalized as follows. (Such generalizations are due to for example [44, 45] in the physics literature, and were previously known as "monads" in the math literature). We consider a complex of holomorphic vector bundles

$$0 \to A \to^a B \to^b C \to 0$$

such that $\ker a = 0$, $\operatorname{im} a$ is a subbundle of B, $\operatorname{im} b = C$ and define our new bundle as

$$E = \ker b / \operatorname{im} a\,.$$

For a hypersurface M in $\mathbb{P}^n$, simple bundles to start with are direct sums of the line bundles $\mathcal{O}(n)$ restricted to M, as in

$$0 \to \oplus\mathcal{O} \to \oplus_{i=1}^m \mathcal{O}(q_i) \to \mathcal{O}\left(\sum_{i=1}^m q_i\right) \to 0\,.$$

This data allows computing the Chern classes:

$$c_n = \left(\sum_{i=1}^m q_i\right)^n - \sum_{i=1}^m q_i^n\,.$$

The dimension of the moduli space can also be computed, but this is not as easy.

A physical realization of this construction is to start with fields λ^i parameterizing sections of B (e.g. the world-sheet fermions of a heterotic string theory), include a superpotential enforcing the constraints $b_i^a \lambda^i = 0$, and gauge invariances identifying $\lambda^i \sim \lambda^i + a^i$. Although it is not the only place this construction appears (e.g. see [46]), the most relevant version for present purposes is in linear $(0, 2)$ models [45]. These constructions have the advantage that they can be studied with conventional world-sheet techniques; a disadvantage is that one requires the anomaly cancellation conditions $c_1 = 0$ and $c_2(V) = c_2(T)$ to get a sensible model, so only a subset of possible V can be obtained.

The anomaly cancellation conditions also appear in D-brane constructions of the dual type-I theories as the consistency condition that the total RR charge vanishes [47]. However in this context we need not consider branes which fill the non-compact dimensions but can instead consider lower dimensional branes, for which these consistency conditions are not required (a point emphasized in [5]). It seems likely that this additional freedom will lead to a simpler theory.

Another construction of vector bundles on a CY is the Serre construction. Given a holomorphic curve (satisfying certain conditions), this produces a rank 2 vector bundle with a section having its zeroes on the curve. In [48] this is used to produce an example of a vector bundle with an obstructed deformation (on a different CICY).

Finally, to conclude this section, there are a few explicit constructions of bundles on $\mathbb{P}^4$ in the literature using monads, such as the Horrocks-Mumford bundle ($r = 2, c_1 = 5, c_2 = 10$) and the bundle of Tango ($r = 3, c_1 = 3, c_2 = 5, c_3 = 5$) [49], which can be restricted to the hypersurface $P = 0$ to produce new examples.

2.3.2 A branes on the quintic

The simplest example of supersymmetric 3-cycles on the quintic are the real surfaces $\operatorname{Im} \omega_j z_j = 0$ with $\omega_j^5 = 1$; this was described in [3] for $\omega = 1$. These cycles are determined by the five phases $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ up to the diagonal $\mathbb{Z}_5$ action $\omega_i \to \omega\omega_i$ (which is just a remnant of the equivalence of homogeneous coordinates under complex multiplication), so they come in a 625-dimensional irrep of the discrete symmetry $S_5 \times \mathbb{Z}_5^4$.

The equation $\sum(\omega_j x_j)^5 = 0$, where $\omega_i x_i \in \mathbb{R}$, always has a unique solution for x_k in terms of the other real coordinates; thus the cycle is the real projective space $\mathbb{R}P^3$. The first homotopy group is $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$; by the discussion above (c.f. [31]) the wrapped 3-branes cannot have any continuous moduli, but they can support a discrete $\mathbb{Z}_2$-valued Wilson line.

To compare these cycles with Gepner boundary states it will be useful to find their intersection matrix. Let us choose the coordinate system $z_1 = 1$ on $\mathbb{P}^4$, so that $\omega_1 = 1$. Regard the cycle $(1, 1, 1, 1, 1)$ as an embedding of the coordinates x_2,x_3 and x_4 into the quintic with positive orientation. The other surfaces are obtained by $\mathbb{Z}_5^4$ rotation from this one, $\prod_{i=1}^5 g_i^{k_i}(1, 1, 1, 1, 1)$. Since the intersection matrix must respect the $\mathbb{Z}_5^4$ symmetry, it can be written as a polynomial in the generators g_i and is determined by the matrix elements

$$\langle (1,1,1,1,1)|(1,\omega_2,\omega_3,\omega_4,\omega_5)\rangle = \langle (1,1,1,1,1)|g_2^{k_2}g_3^{k_3}g_4^{k_4}g_5^{k_5}|(1,1,1,1,1)\rangle, \quad (2.21)$$

where $g_i^{k_i} : z \to \omega^{k_i} z$. S_5 symmetry also constrains the problem in an obvious way.

There are different possibilities for intersections with the surface $(1,1,1,1,1)$ in this coordinate system. If ω_2, ω_3, ω_4 and ω_5 are all different from 1 there is no intersection in this coordinate patch. If only three of them are different from 1 there is exactly one intersection in this coordinate patch and the intersection has the signature sgn $\operatorname{Im}\omega_2 \operatorname{Im}\omega_3 \operatorname{Im}\omega_4$ assuming that $\omega_5 = 1$. If the two surfaces intersect on a higher dimensional locus the intersection number has to be calculated by a small deformation of one of the two surfaces. This deformation has to be normal to both surfaces. Because of the special lagrangian property of the undeformed surfaces this "normal bundle" of the intersection locus can be identified with its tangent space. The intersection number is then given by the number of zeros of a section of the tangent bundle of the intersection locus.

For example, in the case that exactly two ω_j's are not 1 the intersection locus is a circle. A circle can have a nowhere vanishing section of its tangent bundle and the intersection number in this coordinate patch is 0. As another example, let precisely one $\omega_j \neq 1$. The intersection locus is then an $\mathbb{RP}^2$. A section of its tangent bundle has one zero, as can be seen by modding out the 'hedgehog configuration' of an S^2 by $\mathbb{Z}_2$. The orientation of this intersection is given by the intersection in the remaining complex dimension, i.e. by $\operatorname{Im}\omega_j$.

In order to compute the full intersection we must look at all possible patches. This can be done by using the constraint $\prod_{i=1}^5 g_i = 1$ to rewrite $(1,\omega_2,\omega_3,\omega_4,\omega_5)$ as $(\omega_2^{-1}, 1, \omega_2^{-1}\omega_3, \omega_2^{-1}\omega_4, \omega_2^{-1}\omega_5)$ and so on. We then add all of the intersection numbers for all of these patches. Thus, although we find that $\langle (1,1,1,1,1)|(1,\omega,\omega,\omega,\omega)\rangle = 0$ in the $z_1 = 1$ coordinate patch, the total intersection number — the coefficient of $\prod_{i=2}^5 g_i$ in the intersection matrix — is 1. Another example is the intersection of $(1,1,1,1,1)$ with $(1,\omega,\omega,\omega,1)$ which gives a circle in the patch $z_2 = 1$ and a point in the patch $z_1 = 1$.

A simple general formula that matches all of these results is

$$I_{\mathbb{RP}^3} = \prod_{i=1}^{5} (g_i + g_i^2 - g_i^3 - g_i^4). \quad (2.22)$$

3. Stringy geometry

Type-IIB string compactification on a general CY threefold M leads to an $\mathcal{N}=2$, $d=4$ supergravity with $b_{2,1}+1$ vector fields ($b_{2,1}$ vector multiplets plus the graviphoton) and $b_{1,1}+1$ hypermultiplets (including the 4d dilaton); in IIA these identifications are reversed. The most basic physical observables which reflect the structure of M are those described by the special geometry of the vector multiplets. This geometry is determined by a prepotential F_K of Kähler deformations in the IIA case, and by the prepotential F_c for complex structure deformations in the IIB case.

A fundamental result from the study of the worldsheet sigma model is that F_c can be determined entirely from classical target space geometry; it receives no worldsheet quantum (α') corrections. Let us then discuss the complex structure moduli space. Choose a basis for the 3-cycles $\Sigma^i \in H_3(M,\mathbb{Z})$ (where $i = 0,\ldots,b_{2,1}, b_{2,1}+1,\ldots,2b_{2,1}+2$), so that the intersection form $\eta^{ij} = \Sigma^i \cdot \Sigma^j$ takes the canonical form $\eta^{i,j} = \delta_{j,i+b_{2,1}+1}$ for $i = 0,\ldots,b_{2,1}$ (an a cycle with a b cycle). The $b_{2,1}+1$ vector fields come from reducing the RR potential $C^{(4)}$ on the a cycles, while the b cycles produce their $d=4$ electromagnetic duals. Thus a three-brane wrapped about the cycle $\Sigma = \sum_i Q_i \Sigma^i$ has (electric,magnetic) charge vector Q_i. Note that $H_3(X)$ forms a nontrivial vector bundle over the moduli space $\mathcal{M}_c$ of complex structures; a given basis in $H_3(X,\mathbb{Z})$ will have monodromy in $\mathrm{Sp}(b_3,\mathbb{Z})$ as it is transported around singularities in $\mathcal{M}_c$.

The primary observables are the periods of the holomorphic three-form,

$$\Pi^i = \int_{\Sigma^i} \Omega\,.$$

In $\mathcal{N}=2$ language these are the vevs of the scalar fields in the corresponding vector multiplets. The a-cycle Π^i's can be used as projective coordinates on the moduli space; the b-cycle periods then satisfy the relations $\Pi^j = \eta^{ij}\partial\mathcal{F}/\partial\Pi^i$. If we fix (for example) $\Pi^0 = 1$ to pass to inhomogeneous coordinates, the related vector field is the graviphoton. These periods determine the central charge of a three-brane wrapped about the cycle $\Sigma = \sum_i Q_i[\Sigma^i]$:

$$Z = \int_\Sigma \Omega = Q_i \Pi^i\,.$$

Thus the mass of a BPS three-brane is [50]:

$$m_Q = c|Z| = c|Q\cdot\Pi| \tag{3.1}$$

where c is independent of Q. If we use four-dimensional Einstein units for m, it is $c = 1/g_s(\int\Omega\wedge\bar{\Omega})^{1/2}$.

In contrast to F_c, F_K receives world-sheet instanton corrections to the classical computation. The exact worldsheet result can be obtained by mirror symmetry: F_K for IIA on M is equal to F_c for IIB on the mirror W to M. Of course this requires a map between the periods of M and W. This analysis has been carried out for the quintic in [12] (see [51] for a summary) and we will quote the result in this case.

The mirror W to the quintic threefold M can be obtained [52] as a $\mathbb{Z}_5^3$ quotient of a special quintic

$$0 = \sum_{i=1}^{5} z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 \, .$$

The transformation $\psi \to \alpha\psi$ with $\alpha^5 = 1$ can be undone by the coordinate transformation $z_1 \to \alpha^{-1} z_1$ and thus the complex moduli space of W's can be parameterized by ψ^5. This is an "algebraic" coordinate, which although not directly observable, does appear naturally in the world-sheet formulations [53, 54].

The moduli space $\mathcal{M}$ has three singularities, about which the three-cycles in W will undergo monodromy. Each singularity has physical significance. First, $\psi^5 \to \infty$ is the "large complex structure limit" mirror to the large volume limit. In this limit [51]

$$(5\psi)^{-5} \to e^{2\pi i(B+iJ)} \, , \tag{3.2}$$

where B is the NS B-field flux around the 2-cycle forming a basis of $H_2(M)$, and J is the size of that 2-cycle. Next, $\psi^5 \to 1$ is a conifold singularity; here a wrapped three-brane becomes massless [55]. This turns out to be mirror to the "pure" six-brane [56, 57]. Finally, at $\psi^5 = 0$ the model obtains an additional $\mathbb{Z}_5$ global symmetry; this is an orbifold singularity of moduli space. The Gepner model $(3)^5$ lives at this point in Kähler moduli space of M [53].

Each singularity in $\mathcal{M}$ gives a noncontractible loop, which is associated with a monodromy on the basis of 3-cycles in W (or even homology in M) and thus on the periods. We let A be the monodromy induced by $\psi \to \alpha\psi$ around $\psi = 0$; clearly $A^5 = 1$. T will be the monodromy induced by going once around the conifold point, and B will be the monodromy induced by taking $\psi \to \alpha^{-1}\psi$ around infinity. These satisfy the relation $B = AT$. One may make the physics associated with a given singularity manifest by choosing variables (the periods) for which the associated monodromy is simple.

In our case the periods Π_i satisfy a Picard-Fuchs differential equation of hypergeometric type. Since $b^3 = 4$ it is fourth order and quite tractable. There will be four independent solutions and as per the discussion above, we generally want to choose a basis making one of the monodromies simple. Two such bases are particularly natural. The first is the large volume basis which we will denote $(\Pi_6, \Pi_4, \Pi_2, \Pi_0)^t$. Up to an upper triangular transformation this is determined by the asymptotics as $\psi^5 \to \infty$

$$\begin{pmatrix} \Pi_6 \\ \Pi_4 \\ \Pi_2 \\ \Pi_0 \end{pmatrix} \to \begin{pmatrix} -\frac{5}{6}(B+iJ)^3 \\ -\frac{5}{2}(B+iJ)^2 \\ B+iJ \\ 1 \end{pmatrix} . \tag{3.3}$$

The coefficients correspond to the classical volumes of the cycles. The signs were chosen so that the supersymmetric brane configurations have positive relative charges. We will derive the monodromy below.

The other natural basis for us makes the monodromy A simple, and is appropriate for describing the Gepner point. If we choose a solution $\Pi_0^G(\psi)$ analytic near $\psi = 0$, the set of solutions

$$\Pi_i^G(\psi) = \Pi_0^G(\alpha^i \psi) \tag{3.4}$$

will provide a basis with the single linear relation $0 = \sum_{i=0}^{4} \Pi_i^G$. It turns out that the 0-brane period Π_0 (the solution $\tilde{\omega}_0$ of [12, eq. (3.15)]) is analytic near $\psi = 0$ and thus we can set $\Pi_0^G = \Pi_0$ and define the others using (3.4). We then (as in [12]) choose the period vector $(\Pi_2^G, \Pi_1^G, \Pi_0^G, \Pi_4^G)^t$. In this basis, the three monodromy matrices are [5]

$$A^G = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$T^G = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -4 & 1 \end{pmatrix}$$
$$B^G = \begin{pmatrix} -1 & -7 & 5 & -1 \\ 1 & 4 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}. \tag{3.5}$$

In [12], the relation between the large volume and Gepner bases proceeds through a third basis which we will call Π^3, which is naturally described by a particular basis of 3-cycles in W. The intersection form in this basis has the canonical form $\eta_{13} = \eta_{24} = -1$, and the T monodromy is simple: $\Pi_i^3 \to \Pi_i^3 + \delta_{i,2}\Pi_4^3$. Thus Π_4^3 is the vanishing cycle at the conifold and Π_2^3 is its dual. This turns out to be enough information to relate it to the Gepner basis uniquely up to a remaining $\mathrm{SL}(2,\mathbb{Z})$ acting on Π_1^3 and Π_3^3, which we may fix arbitrarily. One then finds a transformation of Π^3 to a basis satisfying (3.3). This is an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the type which was unfixed in the previous step; so the Π^3 basis has no significance intrinsic to our problem of relating Π^G to the large-volume basis. Thus we will merely quote the final result for this change of basis, which is:

$$\Pi = M\Pi^G \qquad Q = Q^G M^{-1} \qquad A = MA^G M^{-1} \ldots$$

[5] There is a typo in [12, table I] as published in Nuclear Physics B.

$$M = L \begin{pmatrix} 0 & -1 & 1 & 0 \\ -\frac{3}{5} & -\frac{1}{5} & \frac{21}{5} & \frac{8}{5} \\ \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.6}$$

Here Q and Q^G are the charge vectors in the large-radius and Gepner basis respectively. (In the notation of [12], $M = KNm$: with K a matrix taking the vector $(Q_4, -Q_6, Q_2, Q_0)$ of their conventions to our conventions; and N taken with $a' = b' = c' = 0$.) The matrix L is an as-yet undetermined $\mathrm{Sp}(4, Z)$ ambiguity in the Q_2 and Q_0 charges of the six- and four-branes:

$$L = \begin{pmatrix} 1 & 0 & -b & -c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with (a, b, c) integers (the (a', b', c') of [12]).

Given the classical intersection form η in the large-radius limit, we can now determine the intersection form in the Gepner basis:

$$\eta_g = M^{-1}\eta(M^{-1})^t = \begin{pmatrix} 0 & -1 & 3 & -3 \\ 1 & 0 & -1 & 3 \\ -3 & 1 & 0 & -1 \\ 3 & -3 & 1 & 0 \end{pmatrix} , \tag{3.7}$$

where $\eta_{14} = -\eta_{41} = -\eta_{23} = \eta_{32} = 1$ from [12].[6] L does not enter since it is symplectic, and so preserves η. η_g has determinant 25 and thus the Gepner basis is not canonically normalized; this point will not be important for us.

We want to better understand the ambiguity L. We can start by comparing the monodromy B with our expectations from the large volume limit. One may define a basis of charges such that Γ_k^{RR} is the charge under the RR potential $C^{(k+1)}$, with the switch in four- and six-brane charge as in (3.3). In this basis the effect of the shift $B \to B + 1$ follows from eq. (2.10):

$$B_L = \begin{pmatrix} 1 & 1 & -\frac{5}{2} & -\frac{5}{6} \\ 0 & 1 & -5 & -\frac{5}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.8}$$

The factors 1/2 and 1/6 in this expression come from expanding the exponential (they can also be seen in (3.3)) and indicate that in this basis the charges are not integers.

[6] The signs $\Sigma_6 \cdot \Sigma_0 = +1$ and $\Sigma_4 \cdot \Sigma_2 = -1$ in the large volume intersection form η follow from the definition (2.12).

The B monodromy in the Π basis (3.3) is

$$B = \begin{pmatrix} 1 & 1 & 3-a & -5-2b \\ 0 & 1 & -5 & -8+a \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.9}$$

Eqs. (3.8) and (3.9) agree if $a = 11/2$ and $b = -25/12$, i.e.if we make a non-integral redefinition of the charge lattice. The explanation of this is that the intersection form in the conventions leading to (3.8) is actually not canonical, because it includes the other terms in (2.10). If we act on the basis (3.3) with the matrix $L(a = 11/2, b = -25/12, c)$, we can see that the charges are modified in precisely this way. The modification due to b comes from the $\hat{A}$ term in (2.12) (as $c_2 = 50$ for the quintic). a induces a two-brane charge on the four-brane and might come from c_1 of its normal bundle. These effects were referred to in [19] as the "geometric Witten effect".

The most interesting ambiguity comes from c which induces zero-brane charge on the six-brane. In [12] this was attributed to the sigma model four-loop R^4 correction in the bulk lagrangian. In the D-brane context, one possibility is that this comes from an as yet unknown term at this order in the D-brane world-volume lagrangians. We should also keep in mind that the intersection form we are computing involves the bulk propagation of the RR fields between the branes, so another possibility is that it comes from a partner to the R^4 term in the bulk lagrangian which affects the RR kinetic term in a curved background.

In [12], the redefinition L was used to make the charge basis integral, but an overall $\mathrm{Sp}(4, \mathbb{Z})$ ambiguity was left over. It is in general more useful to have an integer charge basis so we will follow this procedure (this was already done implicitly as we took integer coefficients in the change of basis). We can resolve most of the $\mathrm{Sp}(4, \mathbb{Z})$ ambiguity by calling the state which becomes massless at the mirror of the conifold point a "pure" six-brane with large volume charges (1 0 0 0), following [56, 57]. This determines $b = c = 0$. A geometrical argument for this is that any fluxes on the six-brane would produce additional contributions to its energy. If there is a line from the large volume limit to the conifold point along which the six-brane becomes massless with no marginal stability issues, this argument will presumably be valid. Another argument is that we will find this state as a Gepner model boundary state with no moduli, as is appropriate for a pure six-brane. Finally, this choice simplifies the charge assignments for the other boundary states.

We still have the ambiguity in a to fix. As it happens this does not enter into the results we discuss, so we have no principled way to do this. We will simply set it to zero.

4. Boundary states in CFT

4.1 Some results from boundary conformal field theory

A CFT on a Riemann surface with boundary requires specifying boundary conditions on the operators. For sigma models these conditions can be derived by imposing Dirichlet and/or Neumann boundary conditions directly on the sigma model fields. For more general CFTs we do not have a nice lagrangian description; so the construction, classification, and interpretation of boundary conditions is not as straightforward. (See [58, 7, 59, 8] and references there for recent work in this direction.)

Given that a CFT has a chiral symmetry algebra, one may simplify the problem by demanding that the boundary conditions are invariant under the symmetry. We can start with the Virasoro algebra which must be preserved (in superstring theory, an $\mathcal{N} = 1$ superconformal symmetry is gauged). Let the boundary be at $z = \bar{z}$ in some local coordinates. Reparameterizations should leave the boundary fixed, so we must impose $T = \bar{T}$. If the remaining symmetry algebra is generated by chiral currents $W^{(r)}$ with spin s_r, then such boundary conditions are

$$W^{(r)} = \Omega \bar{W}^{(r)} \Omega^{\dagger} , \tag{4.1}$$

where Ω is an automorphism of the symmetry algebra.

We are interested in describing BPS D-branes which preserve $\mathcal{N} = 1$ spacetime SUSY. The closed-string sector will have at least $\mathcal{N} = (2, 2)$ worldsheet SUSY and the boundary conditions must preserve a diagonal $\mathcal{N} = 2$ part [60, 61]. Eq. (4.1) leads to two classes of boundary conditions [6]: the “A-type” boundary conditions

$$T = \bar{T} , \qquad J = -\bar{J} , \qquad G^{+} = \pm \bar{G}^{-} , \tag{4.2}$$

and the “B-type” boundary conditions

$$T = \bar{T} , \qquad J = \bar{J} , \qquad G^{+} = \pm \bar{G}^{+} . \tag{4.3}$$

These conventions correspond to the open-string channel where the boundary propagates in worldsheet time. For Calabi-Yau compactification at large volume, A-type boundary conditions correspond to D-branes wrapped around middle-dimensional supersymmetric cycles; and B-type boundary conditions to D-branes wrapped around even-dimensional supersymmetric cycles [6].

A CFT on an annulus can also be studied in the closed-string channel where time flows from the one boundary to the other. The boundaries appear as initial and final conditions on the path integral and are described in the operator formalism by “coherent” boundary states [62, 63]. The boundary conditions (4.1) can be rewritten in the closed-string channel as operator conditions on these boundary states; for example

$$
\begin{aligned}
J_n &= \bar{J}_{-n} \qquad \text{A type} \\
J_n &= -\bar{J}_{-n} \qquad \text{B type.}
\end{aligned} \tag{4.4}
$$

The relative sign change from (4.2), (4.3) can be understood as the result of a $\pi/2$ rotation on the components of the spin one current; it means that the A-type states are charged under (c, c) operators and the B-type under (c, a) operators.

The solution to these conditions [64, 65] are linear combinations of the "Ishibashi states":

$$
|i\rangle\rangle_\Omega = \sum_N |i, N\rangle \otimes U\Omega\overline{|i, N\rangle}. \tag{4.5}
$$

Here $|i\rangle$ is a highest weight state of the extended chiral algebra; the sum is over all descendants of $|i\rangle$; and U is an anti-unitary map with $U\overline{|i,0\rangle} = \overline{|i,0\rangle}^\star$ and $U\bar{W}_n^{(r)}U^\dagger = (-1)^{s_r}\bar{W}_n^{(r)}$.

Modular invariance requires that calculations in either channel have the same result. This gives powerful restrictions on possible boundary states. In particular one requires that a transition amplitude between different boundary states can be written as a sensible open-string partition function, via a modular transformation. For rational CFTs with certain restrictions, Cardy [13] showed that the allowed linear combinations of Ishibashi states (4.5) are:

$$
|I\rangle\rangle_\Omega = \sum_j B_I^j |j\rangle\rangle_\Omega = \sum_j \frac{S_I^j}{\sqrt{S_0^j}} |j\rangle\rangle_\Omega \,. \tag{4.6}
$$

If χ_j is a character of the extended chiral algebra, then S_i^j is the matrix representation of the modular transformation $\tau \to -1/\tau$. In this notation capital and lower-case letters denote the same representation; we use capital letters to denote this particular linear combination of Ishibashi states. We may also associate a bra state to the representation $I^\vee$ conjugate to I:

$$
{}_\Omega\langle\langle I^\vee| = \sum_j {}_\Omega\langle\langle j| B_I^j \,. \tag{4.7}
$$

These boundary states are in one-to-one correspondence with open-string boundary conditions which we will label the same way. Cardy argued that the open-string partition function was determined by the fusion rule coefficients. Let worldsheet time and space be labeled by τ and σ respectively; and let the boundary run from $\sigma = 0$ to $\sigma = \pi$, and the boundary conditions be $I^\vee$ and J, respectively. Then the number of times that the representation k appears in the open-string spectrum is precisely the fusion rule coefficient N_{IJ}^k; in other words, the open-string partition function will be

$$
Z_{I^\vee J} = \sum_k N_{IJ}^k \chi_k \,. \tag{4.8}
$$

4.2 The Gepner model in the bulk

Gepner models [66, 67] (see also [68] for a quick review) are exactly solvable CFTs which correspond to Calabi-Yau compactifications at small radius [53]. They are tensor products of r $\mathcal{N}=2$ minimal models together with an orbifold-like projection that couples the spin structures and allows only odd-integer U(1) charge. We will review their construction here. For simplicity we will discuss theories with $d+r=$ even, where d is the number of complex, transverse, external dimensions in light cone gauge.

Our building blocks are the $\mathcal{N}=2$ minimal models at level k; these are SCFTs with central charge $c=\frac{3k}{k+2}<3$ [69, 70, 71, 72]. The superconformal primaries are labelled by 3 integers, (l,m,s) with

$$0\le l\le k\,;\qquad |m-s|\le l\,;\qquad s\in\{-1,0,1\}\,;\qquad l+m+s=0 \bmod 2\,. \tag{4.9}$$

The integers l and m are familiar from the $\mathrm{SU}(2)_k$ WZW model and can be understood from the parafermionic construction of the minimal models [73, 74]. s determines the spin structure: $s=0$ in the NS sector; and $s=\pm 1$ are the two chiralities in the R sector.[7] The conformal weights and U(1) charges of these primary fields are:

$$\begin{aligned} h^l_{m,s} &= \frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}\,,\\ q^l_{m,s} &= \frac{m}{k+2}-\frac{s}{2}\,. \end{aligned} \tag{4.10}$$

The $\mathcal{N}=2$ chiral primaries are clearly $(l,l,0)$ in the NS sector. The related Ramond sector states $(l,\pm l,\pm 1)$ can be reached by spectral flow. The minimal models can also be described by a Landau-Ginzburg model of a single superfield with superpotential X^{k+2} [75]–[79]. At the conformal point $X^l=(l,l,0)$ and the Landau-Ginzburg fields provide a simple representation of the chiral ring.

The $\mathcal{N}=2$ characters and their modular properties are described in [80, 81, 66, 67]; we will follow the notation in [66, 67]. One extends the s variable to take values in $\mathbb{Z}_4$. The NS characters are labelled by $s=0,2$ and the different values of s denote opposite $\mathbb{Z}_2$ fermion number. The contribution from the NS primary is in $\chi_{l,m,0}$. Similarly, in R sector $s=\pm 1$ denotes contributions from opposite fermion number: the $s=1 (s=3)$ character includes the contribution from the $s=1 (s=-1)$ Ramond-sector primary. These characters are actually defined in the range $l\in\{0,\dots,k\}$, $m\in\mathbb{Z}_{2k+4}$ and $s\in\mathbb{Z}_4$, where $l+m+s=$ even. They obey the identification $\chi^l_{m,s}=\chi^{k-l}_{m+k+2,s+2}$ by which the fields can be brought into the range (4.9).

[7] The variable m in [74], in [66, sec. 2.1], and [67, section 4], is what we are calling $m-s$.

Not every $c = 9$ tensor product of minimal models will give a consistent string compactification with 4d spacetime SUSY. We must find a reasonable GSO projection, and we must project onto states with odd integer U(1) charges [60]. We must then add "twisted" sectors in order to maintain modular invariance. The resulting spectrum is most easily represented by the partition function, for which we require some notation. We will tensor r minimal models at level k_j with the CFT of flat spacetime. The latter also has a $\mathcal{N} = 2$ worldsheet SUSY in our case, and we denote the characters by the indices i. The vector $\lambda = (l_1, \ldots, l_r)$ gives the l_j quantum numbers and the vector $\mu = (m_1, \ldots, m_r; s_1, \ldots, s_r)$, the charges and spin structures. Now define $\beta_{j=1,\ldots,r}$ to be the charge vector with a two at the position of s_j, and all other entries zero; and define β_0 to be the charge vector with all entries one. The modular invariant partition function in light cone gauge can be written as [66, 67]:

$$Z = \sum_{(i,\bar{i}),\lambda,\mu} \sum_{b_0,b_j} \delta_\beta (-1)^{b_0} \chi_{i,\lambda,\mu}(q) \chi_{\bar{i},\lambda,\mu+b_0\beta_0+\sum_j b_j\beta_j}(\bar{q}) \,, \tag{4.11}$$

Here $\chi_{i,\lambda,\mu}$ is the character for the r minimal models specificed by λ, μ and for the character of the flat transverse spacetime coordinates (labelled by i). In the sum, $b_0 = 0, \ldots, 2K-1$, $b_j = 0, 1$ and $K = \mathrm{lcm}\{2, k_j + 2\}$. δ_β is a Kronecker delta function enforcing both odd integral U(1) charge and the condition that all factors of the tensor product have the same spin structure.

The kth minimal model has a $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ symmetry [66, 82] which acts as:

$$g\phi^l_{m,s} = e^{2\pi i \frac{m}{k+2}} \phi^l_{m,s}, \; h\phi^l_{m,s} = (-1)^s \phi^l_{m,s} \,. \tag{4.12}$$

With the above projection, all $\mathbb{Z}_2$ symmetries have the same action on a given state and are identified. The remaining $\mathbb{Z}_2$ symmetry acts only on R states by reversing their sign. The $\mathbb{Z}_{k+2}$ symmetry is correlated with the U(1) charge. In particular, the diagonal generator $G = \prod_j g_j$ is the identity for integral U(1) charges. The Gepner model is an orbifold theory; the orbifold group H is the group generated by G. The remaining discrete symmetry is $\otimes_{i=1}^r \mathbb{Z}_{k_r+2}/H$. For example, the $(k = 3)^5$ model is an orbifold by the diagonal $\mathbb{Z}_5$ of $(\mathbb{Z}_5)^{\otimes 5}$ (and the off-diagonal $\mathbb{Z}_2$ symmetries, to correlate the spin structures).

4.3 Boundary states in the Gepner model

It is difficult to construct the most general boundary state for the Gepner model, because the Gepner model is not rational. Following [7], we will consider states which respect the $\mathcal{N} = 2$ world-sheet algebras of each minimal model factor of the Gepner model separately, and can be found by Cardy's techniques. These might be called "rational boundary states." They are labeled according to Cardy's notation by $\alpha = (L_j, M_j, S_j)$ and an automorphism Ω of the chiral symmetry algebra. In our case there are two choices of Ω giving either A- or B-type boundary conditions; Ω must have the same action on every factor of the tensor product.

Recknagel and Schomerus [7] proved the modular invariance of A- and B-type boundary states with internal part:

$$|\alpha\rangle\rangle = \frac{1}{\kappa_\alpha^\Omega} \sum_{\lambda,\mu} \delta_\beta \delta_\Omega B_\alpha^{\lambda,\mu} |\lambda,\mu\rangle\rangle_\Omega \,. \tag{4.13}$$

The coefficients are:

$$B_\alpha^{\lambda,\mu} = \prod_{j=1}^{r} \frac{1}{\sqrt{\sqrt{2}(k_j+2)}} \frac{\sin(l_j,L_j)_{k_j}}{\sqrt{\sin(l_j,0)_{k_j}}} e^{i\pi\frac{m_j M_j}{k_j+2}} e^{-i\pi\frac{s_j S_j}{2}} \,, \tag{4.14}$$

a result of eq. (4.6) for the minimal models and the extra coefficient κ_α^Ω described in the appendix. Here

$$(l,l')_k = \pi \frac{(l+1)(l'+1)}{k+2} \,.$$

δ_Ω denotes the constraint that the Ishibashi state $|\lambda,\mu\rangle\rangle_\Omega$ must appear in the closed string partition function (4.11). For A-type boundary states this is no constraint as the Ishibashi states are already built on diagonal primary states and δ_β already enforces that total U(1) charge is integral. However, the B-type Ishibashi states have opposite U(1) charge in the holomorphic and antiholomorphic sector, and these only appear as a consequence of the GSO projection; so the δ_B constraint requires that all the m_j are the same modulo k_j+2. Finally, an integer normalization constant C has to be included in κ_α^Ω to get the correct normalization for the open-string partition function.

It is easy to see from eqs. (4.13), (4.14) that the action of the $\mathbb{Z}_{k_j+2}$ ($\mathbb{Z}_2$) symmetries is $M_j \to M_j+2$ ($S_j \to S_j+2$). As a result of the δ_β constraint, the two physically inequivalent choices for S_j are $S = \sum S_j = 0, 2 \bmod 4$. The $S_j =$ odd case seems to be inconsistent because their RR-charges do not fit into a charge lattice together with the $S =$ even states; thus they will violate the charge quantization conditions[8]. In the end, due to the $\mathbb{Z}_2$ symmetry, it is enough to consider only boundary states with $S=0$. A boundary state can be written as

$$\begin{aligned}
&g_1^{\frac{M_1}{2}} \cdots g_r^{\frac{M_r}{2}} h^{\frac{S}{2}} |L_1 \cdots L_r\rangle_\Omega := |L_1 \cdots L_r; M_1 \cdots M_r; S\rangle_\Omega = \\
&g_1^{\frac{M_1-L_1}{2}} \cdots g_r^{\frac{M_r-L_r}{2}} h^{\frac{S}{2}} |L_1 \cdots L_r; M_1' = L_1 \cdots M_r' = L_r; S' = 0\rangle_\Omega \,.
\end{aligned}$$

For B-type boundary states, the δ_β constraint in eq. (4.13) implies in addition that the physically inequivalent choices of M_j can be described by the quantity

$$M = \sum_j \frac{K' M_j}{k_j+2} \,,$$

where $K' = \mathrm{lcm}\{k_j+2\}$.

[8] The amplitude between a $S=$ odd boundary state and a $\tilde{S}=$ even boundary state also has interchanged roles of R- and NS-states in the open string sector.

JHEP08(2000)015

We will be interested in counting the number of moduli for a D-brane state; these will be the massless bosonic (i.e. NS) open-string states. To find their contribution to the open-string partition function, it is enough to examine the NS-NS part of a transition amplitude in the internal dimensions. The reason is that the (open-string) NS characters arising from the modular transformations of the RR part of the transition amplitude come with an insertion of $(-1)^F$ [80, 81]. With this in mind, a calculation similar to that in [7] leads to[9]

$$Z^A_{\alpha\tilde{\alpha}}(q) = \frac{1}{C}\sum_{\lambda',\mu'}^{NS}\sum_{\nu_0=0}^{K-1}\prod_{j=1}^{r} N^{l'_j}_{L_j,\tilde{L}_j}\,\delta^{(2k_j+4)}_{2\nu_0+M_j-\tilde{M}_j+m'_j}\,\chi^{\lambda'}_{\mu'}(q)\,, \tag{4.15}$$

and

$$Z^B_{\alpha\tilde{\alpha}}(q) = \frac{1}{C}\sum_{\lambda',\mu'}^{NS}\delta^{(K')}_{\frac{M-\tilde{M}}{2}+\sum\frac{K'}{2k_j+4}m'_j}\prod_{j=1}^{r} N^{l'_j}_{L_j,\tilde{L}_j}\,\chi^{\lambda'}_{\mu'}(q)\,. \tag{4.16}$$

(Here $\delta^{(n)}_x$ is one when $x = 0 \bmod n$ and zero otherwise.) This shows that only a U(1) projection and the $\mathrm{SU}(2)_k$ fusion rule coefficients constrain the open string spectrum of B-type boundary states; these states are much richer as a consequence.

The condition that two D-brane boundary states $|\alpha\rangle\rangle$ and $|\tilde{\alpha}\rangle\rangle$, with the same external part, preserve the same supersymmetries is [7]:

$$Q(\alpha-\tilde{\alpha}) := -\frac{S-\tilde{S}}{2}+\sum_{j=1}^{r}\frac{M_j-\tilde{M}_j}{k_j+2} = \text{even}\,. \tag{4.17}$$

To explore the charge lattice of the boundary states, and to find the geometric interpretation of given boundary states, we wish to calculate the intersection (2.11), (2.12) of our branes. The CFT quantity which computes this is $I_\Omega = \mathrm{tr}_R(-1)^F$ in the open string sector [11]. The best way to do this is to start in the closed string sector and to do a modular transformation to the open string sector. In the closed string sector this trace corresponds to the amplitude between the RR parts of the boundary states with a $(-1)^{F_L}$ inserted. The calculation is done in the Appendix and the result for A-type boundary states is:

$$I_A = \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}}\sum_{\nu_0=0}^{K-1}\prod_{j=1}^{r} N^{2\nu_0+M_j-\tilde{M}_j}_{L_j,\tilde{L}_j}\,. \tag{4.18}$$

For B-type boundary states,

$$I_B = \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}}\sum_{m'_j}\delta^{(K')}_{\frac{M-\tilde{M}}{2}+\sum\frac{K'}{2k_j+4}(m'_j+1)}\prod_{j=1}^{r} N^{m'_j-1}_{L_j,\tilde{L}_j}\,. \tag{4.19}$$

[9] $N^l_{L,\tilde{L}}$ are the $\mathrm{SU}(2)_k$ fusion rule coefficients [83]: they are one if $|L-\tilde{L}| \le l \le \min\{L+\tilde{L}, 2k-L-\tilde{L}\}$ and $l+L+\tilde{L} = $ even, and zero otherwise; note that our indices thus differ from those in [83] by a factor of two.

The intersection matrix depends only on the differences $M - \tilde{M}$ as was required by the discrete symmetry. We also see that the $\mathbb{Z}_2$ action $S \to S + 2$ changes the orientation of a brane.

In the next section we will rewrite these formulas in a more compact notation and use them to identify the charges of the boundary states.

4.4 D-branes on K3 and the Mukai formula

For compactifications with $\mathcal{N} = 4$ worldsheet supersymmetry, the index in the Ramond sector is directly related to the number of marginal operators in the NS sector. We now use this to give a CFT proof of Mukai's formula [84, 19] for the dimension of the moduli space of 1/2-BPS D-brane states.

K3 compactifications are geometric throughout their moduli space [85]. The BPS D-brane states in these compactifications are described by coherent semistable sheaves E [19] which can be labelled by the Mukai vector [84, 19]. In terms of the rank r and Chern classes c_i of E, this is

$$v(E) = \big(r, c_1, \tfrac{1}{2}c_1^2 - c_2 + r\big) \in H^0(M, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(M, \mathbb{Z}). \tag{4.20}$$

There is a natural inner product on the space of Mukai vectors:

$$\langle (r, s, \ell), (r', s', \ell') \rangle = s \cdot s' - r\ell' - \ell r' , \tag{4.21}$$

where $s \cdot s'$ is defined by the natural intersection pairing of 2-cycles on M. In fact this is just (minus) the intersection form (2.12).

Mukai's theorem [84] states that the complex dimension of the moduli space of an irreducible coherent sheaf E is:

$$\text{dimension} = \langle v(E), v(E) \rangle + 2 \, . \tag{4.22}$$

We now argue that this follows from the relation

$$\operatorname*{tr}_{a,a} (-1)^F = \langle v(E_a), v(E_a) \rangle \tag{4.23}$$

and general properties of supersymmetry. First, only two $d = 2$, $\mathcal{N} = 4$ representations have nonvanishing Witten indices [86, 87]. We list them below together with the NS weights related by spectral flow:

$$\begin{aligned} &\text{identity rep.} : (h = 0, \ell = 0)_{\mathrm{NS}} \longrightarrow \left(h = \frac{1}{4}, \ell = \frac{1}{2}\right)_{\mathrm{R}} \qquad \mathrm{tr}(-1)^F = -2 \\ &\text{“massless” rep.} : \ (h = \frac{1}{2}, \ell = \frac{1}{2})_{\mathrm{NS}} \longrightarrow \left(h = \frac{1}{4}, \ell = 0\right)_{\mathrm{R}} \qquad \mathrm{tr}(-1)^F = 1 \, , \end{aligned} \tag{4.24}$$

where ℓ is the $\mathrm{SU}(2)_R$ isospin. The identity representations lead to world-volume $d = 6$, $\mathcal{N} = 1$ (or $d = 4$, $\mathcal{N} = 2$) gauge multiplets, while the massless representations lead to world-volume half-hypermultiplets, so there will be one complex scalar in the open-string sector for each massless multiplet.

Let there be N_g identity and N_m massless multiplets; then the Witten index is

$$\mathrm{tr}(-1)^F = N_m - 2N_g. \tag{4.25}$$

Using (4.23) we find that (4.22) will be true if the world-volume theory has a (Higgs branch) moduli space of complex dimension $N_m - 2N_g + 2$. This moduli space is essentially determined by the $d = 6$, $\mathcal{N} = 1$ world-volume supersymmetry: it is the hyperkähler quotient of the configuration space by the subgroup G of the gauge group which acts non-trivially on the hypermultiplets. The resulting space has complex dimension $N_m - 2\mathrm{dim}G$.

Now, any brane configuration will have an overall U(1) acting trivially whose partners in the vector multiplet are the center of mass position of the brane; if more U(1)s act trivially we will have more center of mass moduli, so such a configuration must correspond to a reducible bundle. Therefore $\mathrm{dim}G = N_g - 1$ for an irreducible bundle and we have proven (4.22).

4.5 Generalizations

Mukai's theorem used the Hirzebruch-Riemann-Roch formula together with special properties of K3 surfaces; these properties allowed one to extract the dimension of the moduli space of a bundle directly from the holomorphic Euler characteristic. We have a similar statement for CY threefolds if we keep track of both chiralities separately. The self-intersection number of a brane on a threefold is of course zero, but we can get non-trivial statements if we consider the intersection of two different branes.

For example, consider the index of the Dirac operator on the bundle E. Since the world-volume is Kähler this is

$$\mathrm{ind}\, \not{D} = \sum_{i=0}^{3} (-1)^i \mathrm{dim} H^i(M, E) = \chi(E)$$

which is the holomorphic Euler characteristic. By the Hirzebruch-Riemann-Roch formula,

$$\chi(E) = \int_M \mathrm{ch}(E)\mathrm{Td}(TM). \tag{4.26}$$

Here

$$\mathrm{ch}(E) = r + c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2c_2(E)\right) + \frac{1}{6}\left(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)\right) + \cdots,$$

and

$$\mathrm{Td}(TM) = 1 + \frac{c_2(TM)}{12} + \cdots = 1 - \frac{p_1(TM)}{24} + \cdots.$$

Thus on a threefold, $\mathrm{Td}(M) = \hat{A}(TM)$, and combining eqs. (4.26) and (2.12), we find:

$$\mathrm{ind}\, \not{D} = \langle D6, D(E) \rangle = \operatorname*{tr}_{D6,D(E)} (-1)^F , \tag{4.27}$$

where $D(E)$ is the D-brane representation or generalized Mukai vector for E.

On the other hand, the Ramond ground states which contribute to the open string index are exactly the fermion zero modes which contribute to the index of $\not{D}$. In the type-I case where E is a gauge bundle with vevs entirely in an SU(3) subgroup and with the gauge connection equal to the spin connection, $c_1(E) = 0$; this gives a brane picture of the standard result

$$N_g = (\#\text{ of generations}) = \int_M \frac{c_3}{2}$$

for this case. If we are interested not in the bulk gauge theory on 9-branes in type I but in a gauge theory on a brane B intersecting another brane A, the generalization is that the number of generations (with respect to the B gauge group) associated with the brane A is the intersection form $\langle A, B \rangle$. For B-type branes this follows from eq. (2.12) and the Hirzebruch-Riemann-Roch theorem for the bundle $E(A)^* \otimes E(B)$; for A-type branes each intersection contributes a chiral multiplet with chirality given by the sign of the intersection [17].

5. Discussion of the 3^5 model

Let us apply these results to the example to model $(k = 3)^5$, the Gepner point in the moduli space of the quintic. We will consider boundary states labelled by $L_j \in \{0, 1\}$, $0 \le M_j < (2k + 4) = 10$, and $S = 0$. Let the $\mathbb{Z}_5^4$ symmetry be generated by the operators g_j taking $M_j \to M_j + 2$, and satisfying $g_1 \cdots g_5 = 1$. Note that $g_j^{1/2}$ which takes $M_j \to M_j + 1$ is well-defined for these states (using the identifications on LMS, it relates branes to antibranes).

We will be particularly interested in computing the intersection forms (4.18) and (4.19), as we will be able to use them to extract the charges and open string spectrum for a given brane. The main advantage of considering these quantities over the charges themselves is that they are canonically normalized, as already noted in [1].

We can consider the intersection form as a matrix I acting on the space of boundary states; since it commutes with $\mathbb{Z}_5^4$ it can be written as a function of the generators g_i. The main content of formulae (4.18) and (4.19) is contained in the SU(2) fusion rule coefficients. In these equations the labels M_j, $\widetilde{M}_j$ can be thought of as indices of a matrix acting on the states. The particular fusion coefficients we will need are:[10]

$$\begin{aligned}
N_{00}^{M_j-\widetilde{M}_j} &\to (1-g_j^4)\,,\\
N_{01}^{M_j-\widetilde{M}_j} &\to g_j^{\frac{1}{2}}(1-g_j^3) = N_{00}\ g_j^{\frac{1}{2}}(1+g_j^4)\,;\\
N_{11}^{M_j-\widetilde{M}_j} &\to (1+g_j-g_j^3-g_j^4) = N_{01} g_j^{\frac{1}{2}}(1+g_j^4)\,.
\end{aligned} \tag{5.1}$$

These various fusion matrices are related by successive multiplication with $g_j^{\frac{1}{2}}(1+g_j^4)$, so we can express the RR charges of all our boundary states in terms of those for $Q(|00000\rangle_\Omega)$.

By eq. (4.17) there are two cases of pairs of branes preserving a common susy. If the total ΔL is even (so integral powers of g appear), a pair with $\Delta M = \Delta S = 0$ (brane and brane) will preserve susy. If the total ΔL is odd (powers $g^{5k+5/2}$ appear), a pair with $\Delta M = 5$ and $\Delta S = 2$ (brane and anti-brane) will preserve susy.

In the case that the two D-branes are both A-type or B-type, the massless open string spectrum can also be expressed in terms of the fusion coefficients. It is easy to see from (4.15) and (4.16) that if the two boundary states are the same, there is exactly one vacuum and one spectral flow operator in the open string channel; if they are not the same, neither state propagates. This means that the unbroken worldvolume gauge group is (the center-of-mass) U(1), and the brane can be viewed as a single object (a priori, it still might be a bound state).

The SUSY-preserving moduli of the D-branes are constructed from chiral vertex operators. The Witten index counts these operators albeit with a sign depending on their chirality. In our explicit CFT calculation we can remove this sign by hand, and thus the total number of chiral fields can be calculated using (4.18) and (4.19) with the fusion matrices replaced by their absolute values.[11] We can again write this "modified" matrix as a polynomial $P_\Omega(g_j)$ in the shift matrices g_j. For example, the matrix for boundary states $|11111\rangle_B$ is:

$$P_B(g) = (1+g+g^3+g^4)^5\,. \tag{5.2}$$

If spacetime supersymmetry is preserved, the chiral fields have integer U(1) charges, and are related to antichiral fields by spectral flow. In particular charge-2 chiral fields in $Z^\Omega_{\alpha\tilde{\alpha}}$, are related to charge-$-1$ antichiral fields in $Z^\Omega_{\alpha\tilde{\alpha}}$; the latter are the hermitean conjugate of charge-1 chiral fields in $Z^\Omega_{\tilde{\alpha}\alpha}$. Thus $\sum_k m_k$ in the open-string channel will be a multiple of 5 for marginal, chiral vertex operators. Examination of the fusion coefficients in (4.18) and (4.19) reveals that the number of massless chiral superfields is given by counting terms in

$$\frac{1}{2}(P_\Omega(g_j)-2)$$

with the total power of g being a multiple of 5/2.

[10] The coefficients for $m > l$ are defined in the Appendix.

[11] In other words, we define $N^m_{L\tilde{L}} = +N^{-m-2}_{L\tilde{L}}$, rather than the opposite sign in the appendix.

Applying these statements to eq. (5.2) shows that the D-brane described by $|11111\rangle_B$ has 101 marginal operators. This particular case can also be worked out by checking that the fusion rules lead to all possible L values, so for every operator in the (c, c) ring of the model there is a corresponding chiral open string operator.

5.1 A boundary states

The intersection matrix (4.18) for the A-type boundary states with $L_j = 0$ is

$$I_A = (1 - g_1^4)(1 - g_2^4)(1 - g_3^4)(1 - g_4^4)(1 - g_1 g_2 g_3 g_4). \tag{5.3}$$

To determine the rank of the intersection matrix we can count the number of nonzero eigenvalues. The g_j can be diagonalized as $g_j = \mathrm{diag}(1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}})$. Zero eigenvalues appear if a $g_j = 1$ or if $g_1 g_2 g_3 g_4 = 1$. The combinatorics leads to 204 nonzero eigenvalues, which is the number of independent 3-cycles on the quintic. Thus, the $L_j = 0$ states provide a basis for the charge lattice. So far as we can tell they do not provide an integral basis of the charge lattice. Furthermore, the charges of the other A-type Gepner boundary states can be obtained from these by successive multiplication by $g_j^{\frac{1}{2}}(1+g_j^4)$; for example, $Q(g_1^{\frac{1}{2}}|10000\rangle_A) = Q(|00000\rangle_A) + Q(g_1|00000\rangle_A)$, so these are even farther from an integral basis.

The intersection matrix for the $|11111\rangle_A$ states,

$$\prod_{i=1}^{5}(1 + g_i - g_i^3 - g_i^4)$$

coincides with the intersection matrix (2.22) for the three-cycles $\mathrm{Im}\,\omega_j z_j = 0$, and thus we identify these states with the $\mathbb{RP}^3$'s.

This leads to a potential contradiction with the large volume limit in that the $L = 1$ states have one marginal operator, while the $\mathbb{RP}^3$'s do not. Although it might be that this is indeed a contradiction, from what we know at present an equally likely resolution is that the $L = 1$ marginal operator is not strictly marginal; in other words the world-volume theory has a superpotential for the corresponding field ψ, perhaps of the form

$$W = \psi^3 + \psi\phi$$

where ϕ is the Kähler modulus (ψ^5 in the notation of section 3). Such a superpotential has two ground states and would also fit the fact that the $\mathbb{RP}^3$ has a $\mathbb{Z}_2$ Wilson line in the large volume limit.[12]

[12](Note added in v2): Actually, the two choices of Wilson line are topologically distinct bundles so they would not be continuously connected in the large volume limit. This would suggest that the potential should have a unique minimum. On the other hand, it can be shown that any simply connected six-dimensional manifold X with $H^*(X)$ torsion-free (such as the quintic CY) has $K(X) \cong H^*(X)$, and thus the K theory class distinguishing the two bundles becomes trivial

JHEP08(2000)015

5.2 B boundary states

As we discussed in the previous section, the B-type boundary states at fixed L_j are described by the single integer, $M = \sum M_j$ and the g_j for different j are identified. The intersection matrix (4.19) for $L = 0$ states can be written as:

$$I_B = (1 - g^{-1})^5 = 5g - 10g^2 + 10g^3 - 5g^4. \tag{5.4}$$

We want to describe these boundary states in the Gepner basis. The Gepner intersection form (3.7) in the same notation is:

$$I_g = -g + 3g^2 - 3g^3 + g^4\,. \tag{5.5}$$

A linear change of basis preserving the action of $\mathbb{Z}_5$ can be written as a polynomial in the operator g as well and a transformation of the form $I \to mIm^t$ will be $I \to Im(g)m(g^{-1})$. The relation

$$I_B = (1 - g)(1 - g^{-1})I_g$$

provides this change of basis.

The results of section 3 allow us to write these charges in the large volume basis. The Gepner charge vector Q_G is related to the large volume charge vector Q as

$$Q = Q_G M^{-1}\,.$$

Thus $Q_G = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$ becomes $Q = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}$ which is a pure (anti)six-brane. The other charges can be found by acting with the operator A_L.

One can now compute the charges for the $L \neq 0$ branes by using the multiplicative relation in (5.1). For example, we have

$$Q(g^{\frac{5}{2}}h|10000\rangle_B) = -Q(g^2|00000\rangle_B) - Q(g^3|00000\rangle_B)\,.$$

Starting with $M = 0$ and successively applying this operation produces a subset of branes which preserve the same supersymmetry. This can be checked by computing the central charges using the periods at the Gepner point, which are simply the fifth roots of unity. Thus the central charge for the L'th brane in this series is

$$Z(L) = \left(2\cos\frac{\pi}{5}\right)^L Z(0)\,.$$

The charges in the Gepner basis charges can written in large volume basis viq eq. (3.6). Tabulating these results and the numbers of marginal operators, we have (for the $\mathbb{Z}_5$ representatives related to the six-brane) table 1.

when lifted to the CY. (We thank D. Freed and J. Morgan for explaining this to us.) Thus there is no candidate for a space-time topological charge which could distinguish the two D-branes, and it is not ruled out that transitions between the two choices of bundle are possible in the full string theory.

L	Q_6	Q_4	Q_2	Q_0	dim
00000	-1	0	0	0	0
10000	2	0	5	0	4
11000	1	0	5	0	11
11100	3	0	10	0	24
11110	4	0	15	0	50
11111	7	0	25	0	101

Table 1: Charges and numbers of marginal operators of the D-branes.

The simple pattern $Q_{L+1} = Q_L + Q_{L-1}$ follows from the identity $(-g^2 - g^3)^2 = 1 - g^2 - g^3$.

It is also easy to compute the number of marginal operators between pairs of distinct boundary states. For example, $|00000\rangle_B$ and $|(1\ldots)^L(0\ldots)\rangle$ have (for $1 \leq L \leq 5$) $4, 3, 3, 4$ and 1 (respectively) marginal operators. Each corresponds to a chiral superfield of charge $(1, -1)$ and its charge conjugate (since the mutual intersection numbers are zero, none of these pairs has chiral spectra). The number of operators between two branes of higher L of course depends on which L_i are non-zero.

5.3 Comparison with geometrical results

To what extent can we compare these results with the geometrical branes and bundles we discussed in section 2? The only clear match is the six-brane which indeed has no moduli as expected.

Our states can plausibly be identified with vector bundles since they obey the stability condition $c_2 > 0$. We were not able to identify any of them with the explicit constructions we mentioned in section 2. This may just reflect our lack of knowledge of vector bundles on the quintic; thus we might regard our results as predictions of the existence of new vector bundles. We should note that the numbers of marginal operators we obtained are only upper bounds for the dimension of the moduli space as in general these theories will have potentials.

The problematic objects are the $|11000\rangle_B$ branes as an object with these charges cannot be a classical line bundle. For reasons explained in section 2 we do not believe it is a quantum bound state either, since we have found it at string tree level. There is a piece of evidence that it is some sort of bound state of the six-brane with the two-brane (2.19): namely, they come in the same multiplet of the discrete symmetries. Like all B branes, the $|11000\rangle_B$ branes are invariant under $\mathbb{Z}_5^4$, while S_5 acts by permuting the L_i labels. The two-brane construction (2.19) also picks out two of the five coordinates and thus comes in the same multiplet. This identification creates a puzzle opposite to the one we faced for the $\mathbb{RP}^3$'s: the geometric object appears to have more moduli (12) than the boundary state. Such a mismatch could not be fixed by a superpotential. On the other hand, it could be that the (unknown) mechanism which binds the two-brane to the six-brane removes moduli, so this is not a clear disagreement.

One candidate for such a bound state is the instanton in noncommutative U(1) gauge theory [88]. Again by analogy with flat space, (since noncommutative gauge theory has not been formulated on curved spaces, this is all we can say), at generic values of B we might expect the D6-brane gauge theory to be noncommutative [89, 90, 91]. The center-of-mass position of the instanton would then (presumably) give the moduli of a two-brane and provide at least some of the moduli we observe. A potential problem with this idea is that we can continue to $B = 0$ in the large volume limit, and there is no sign that this bound state is unstable there.

One may ask why the D0-brane does not appear on our list. One possible explanation is that the path from the large volume limit to the Gepner point crosses a line of marginal stability, and the D0 does not exist at the Gepner point. To test this we found the periods for all the branes in table 1 by numerically integrating the Picard-Fuchs equations along the negative real ψ axis. We found that the D0 is lighter than any brane from the list along the whole trajectory, so we have no evidence for instability. Our favored explanation is simply that all of the B branes by construction are invariant under the $\mathbb{Z}_5^4$ discrete symmetry, while any location we might pick for the D0 would break some of this symmetry. Thus, even if the D0 exists at the Gepner point, it cannot be a rational boundary state, at least in this model.

6. Superpotential and topological sigma models

The calculations of the previous section describe the field content of the D-brane world-volumes, but not their dynamics. The primary question in this regard is to find the world-volume potential and true moduli spaces for the brane theories. In CFT language, the marginal boundary operators operators we found might not be strictly marginal.

$\mathcal{N} = 1$, $d = 4$ supersymmetry tells us that the world-volume potential will be a sum of F-terms and Fayet-Iliopoulos D-terms. The D-terms are simply determined by the gauge group and charges of the matter fields. In the case of a single brane or N identical branes we have checked in the models we are studying that the gauge group is U(N) with all matter uncharged under the diagonal U(1), so there is no possibility for a D-term. More generally we must consider such terms, for example in the case of D0-branes near orbifold points.

However, we may expect a non-vanishing superpotential, in general constrained only by holomorphy and the symmetries of the problem. These conditions are often stronger than they might appear, but in general the superpotential must be found by explicit computation. It should eventually be possible to do exact calculations at the Gepner point, as we will discuss in the next section. In this section we will try to make some general statements about the superpotential in these models by showing that they can be calculated as amplitudes in some topologically twisted version of the open string theory. In particular we will use this fact to describe the cubic term in the superpotential, and to discuss to what extent the superpotential couples to the background CY geometry.

6.1 Known examples of brane superpotentials

In order to motivate the search for superpotentials in these theories we will start with a few examples where we know they arise. The most obvious example is N D3-branes in flat space; one may write the $N = 4$ lagrangian in $N = 1$ notation so that there are 3 adjoint complex scalar fields $Z^{i=1,2,3} = Z^i_a t^a$ with the superpotential $\mathrm{Tr}\, Z^1[Z^2, Z^3]$ (here t^a are adjoint matrices for $\mathrm{U}(N)$). Of course this vanishes for $N = 1$ but not for $N > 1$.

A plausible generalization of this to weak curvature (still preserving $N = 1$ world-volume SUSY) is a function W written as a single trace of the adjoint chiral superfields and with the property that

$$\frac{\delta}{\delta Z^i_a}\frac{\delta}{\delta Z^j_b}\frac{\delta}{\delta Z^k_c}W = f^{abc}\Omega_{ijk}(z) \tag{6.1}$$

for variations around the diagonal vevs $Z^i = z^i\mathbf{1}$. The assumption of this form of the superpotential is a fairly weak constraint; see [92] for analysis along these lines. In the case of a large Calabi-Yau threefold, we will find below that this assumption is correct, and that Ω is the holomorphic $(3, 0)$ form of the threefold.

A well-studied genuinely stringy example is that of D-branes near orbifold singularities, or near resolved orbifolds with string-scale curvature. In these examples a "single" brane (in the orbifold limit they are described by Chan-Paton factors in the regular representation of the orbifold group) can have a superpotential, which furthermore can have non-trivial dependence on the closed string moduli. A "single" brane is described via Chan-Paton factors transforming in the regular representation of the orbifold group [5]. The superpotential takes the general form

$$W = \mathrm{Tr}\, Z^1[Z^2, Z^3]|_{proj} + \zeta_i\, \mathrm{Tr}\, Z_i\,. \tag{6.2}$$

The spectrum of these models is obtained as a subset of the $\mathcal{N} = 4$ SYM spectrum [5, 93, 94], and the notation "$W|_{proj}$" indicates that the $\mathcal{N} = 4$ superpotential is simply restricted to this subset. The ζ_i are closed string moduli.

This intrinsically stringy background illustrates the important lesson that by varying both closed and open string moduli, it is possible to bring down new massless open string states invisible in the weakly-curved geometric limit described above. For example, the $\mathbb{C}^2/\mathbb{Z}_2$ model of a single brane has $\mathrm{U}(1)$ gauge symmetry generically; but when both closed- and open-string moduli are tuned to the orbifold point, the gauge symmetry is enhanced to $\mathrm{U}(1)^2$. Furthermore a new branch of moduli space meets this point, where the single brane breaks up into branes wrapping the shrunken cycles of the orbifold [5, 95]. This new branch is a transition from Higgs to Coulomb branch and as usual in supersymmetric gauge theory it is almost impossible to predict such transitions starting from the Higgs branch. In the present context we see that we should be wary of arguments that rely on the distinction between configurations involving "one" or "several" branes, or equivalently "one" or "several" distinct world-sheet boundary conditions, as they can be continuously connected. Other examples where this distinction is questionable are a small instanton leaving a Dp-brane as a D$p-4$-brane, or an intersection of 2-branes as described by [96].

Another point we will return to is that the closed string moduli ζ_i which appear in the superpotential in this example are complex structure moduli. Of course orbifold resolution also depends on Kähler moduli, but these enter in the Fayet-Iliopoulos D-terms.

Our final example is the superpotential on a wrapped two-brane. Recall that a supersymmetric theory arises when we wrap the 2-brane on a holomorphic cycle. The massless fields correspond to infinitesimal deformations of this cycle into a cycle close by in the D-brane moduli space. Witten [97] has argued that an M-theory two-brane (or 5-brane) wrapped around a two-cycle Σ has the superpotential

$$W(\Sigma) = \int_B \Omega \tag{6.3}$$

when Σ is homologically trivial, where B is a three-manifold bounded by Σ. Indeed, this is a holomorphic functional of the embedding coordinates, which is stationary by holomorphic curves. When Σ is in a nontrivial homology class, the superpotential is defined up to an additive constant as:

$$W(\Sigma) - W(\Sigma_0) = \int_B \Omega\,, \tag{6.4}$$

where Σ_0 is an arbitrarily chosen referent holomorphic 2-cycle in the same homology class as Σ, and B has boundary $\Sigma - \Sigma_0$. Here the additive constant depends both on Σ_0 and the homology class of B. For purely classical, geometric deformations these formulae should hold for D2-branes; there may also be terms arising from the gauge fields on the D2-brane worldvolume.

Before discussing the computation in general, we note that in all of our examples, which are of B-type branes, the superpotential depends on closed string moduli only through the complex structure, not through the Kähler structure. Could it be that this is a general statement?

We can see some potential problems with the statement by considering the other branes on the list. First of all, we need to describe not just the embedding but also the gauge bundle on the branes. To the extent that this is determined by a choice of a holomorphic vector bundle, this will fit into the same class of problems depending only on complex structure data. However, one might object that general four- and six-brane configurations involve a gauge bundle with $c_2 \neq 0$ and such a holomorphic bundle will correspond to a solution of Yang-Mills only if it is stable, a condition which depends on the Kähler class. This condition indeed should enter into the potential but as we discussed in section 2, it is more natural to expect that it appears as a D-term, which would not contradict the decoupling statement.

Mirror symmetry for boundary states [6] and the statement we are considering would together imply that the superpotential for A-type branes depends on the Kähler structure of the background, but not the complex structure. But any formula for the potential on the brane analogous to (6.3) will necessarily involve both structures, since the special lagrangian condition cannot be stated without bringing in Ω.

This situation can only be compatible with our decoupling statement if the terms involving Ω are D-terms, an assertion not contradicted by any existing results.[13] One might object that this possibility would require charged matter under a gauge group which is not immediately apparent, but the small instanton and orbifold examples show that such gauge groups can be broken and become invisible in the large volume limit. This would lead to the further interesting possibility that, at special moduli points in the space of a "single" 3-brane, enhanced gauge symmetry could appear. The simplest way this could happen is for the brane to split in two at a self-intersection, leading to $\mathrm{U}(1)^2$ gauge symmetry.

We conclude that we have several examples in which decoupling (before taking stringy corrections into account) is clear, and no examples in which it is clearly false. Thus we will consider this decoupling statement further below.

6.2 CFT computation of superpotential

Given the above examples, we have good reason to believe that the Gepner model boundary states we have constructed correspond to D-branes with worldvolume superpotentials. We want to know how to calculate these in the models at hand.

We are interested in BPS D-branes in $\mathcal{N}=2$ compactifications of type-II string theory; these lead to $\mathcal{N}=1$ worldvolume theories. Thus the open- (closed-) string sectors will have $\mathcal{N}=2$ ($\mathcal{N}=(2,2)$) worldsheet supersymmetry [60, 61]. To fix notation we write out the OPE algebra for the holomorphic piece:

$$\begin{aligned}
T(z)T(w) &\sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \\
T(z)G^{\pm}(w) &\sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{(z-w)} + \cdots \\
G^{\pm}(z)G^{\pm}(w) &\sim \cdots
\end{aligned} \tag{6.5}$$

[13]Related questions are being considered by G. Tian.

$$
\begin{aligned}
G^+(z)G^-(w) &\sim \frac{\frac{c}{6}}{(z-w)^3} + \frac{\frac{1}{2}J(w)}{(z-w)^2} + \frac{1}{2}\frac{T(w)+\frac{1}{2}\partial J(w)}{(z-w)} + \cdots \\
J(z)G^\pm(w) &\sim \pm\frac{G^\pm(w)}{(z-w)} + \cdots \\
J(z)J(w) &\sim \frac{\frac{c}{3}}{(z-w)^2} + \cdots .
\end{aligned}
$$

For compactifications with $c = 9$ $J(z)$ can be constructed from the internal part of the spacetime SUSY current [60]. It can be written in terms of a single boson H,

$$
J(z) = i\sqrt{3}\partial H \tag{6.6}
$$

and operators with charge q under this U(1) R-symmetry can be written as:

$$
\mathcal{O}_q = e^{i(qH/\sqrt{3})}\mathcal{O}_0\,. \tag{6.7}
$$

The spacetime SUSY currents can be constructed from the macroscopic spin fields and the internal U(1) current algebra. In the ($\pm 1/2$) picture, the currents can be written as:

$$
Q_{\pm\frac{1}{2},\alpha}(z) = e^{\pm\phi/2}S_\alpha\Sigma^\pm \tag{6.8}
$$

where

$$
\Sigma^\pm = e^{\pm i\sqrt{3}H/2} \tag{6.9}
$$

is the spectral flow operator of the $\mathcal{N} = 2$ worldsheet algebra, mapping the NS sector to the R sector and vice-versa. ϕ is the bosonized superconformal ghost.

On the 4d noncompact worldvolume, we can have massless chiral superfields Φ^i_{IJ} with scalar components ϕ^i, fermionic components ψ^i and auxiliary components F^i. i will label the (complex) internal moduli of the D-brane configuration on the CY threefold M; these moduli correspond to marginal boundary operators of the internal CFT. (IJ) label gauge indices, which are described by Chan-Paton factors on the worldsheet. These could be adjoint indices if there are coincident branes, or bifundamental indices if there are several types of (possibly intersecting) branes. More abstractly, the off-diagonal terms are boundary condition-changing operators [13] and the diagonal terms are boundary condition-preserving operators.

The superpotential can be written via a holomorphic function $W(\Phi)$ and it contributes the following terms to the lagrangian:

$$
\begin{aligned}
&\int d^4x \left(d^2\theta \;\; \mathrm{tr}\, W(\Phi) + \text{h.c.}\;\right) = \\
&\qquad = \int d^4x \left(\partial_i\partial_j\frac{\partial}{\partial\phi^i_{IJ}}W(\phi)F^i_{IJ} - \frac{\partial}{\partial\phi^i_{IJ}}\frac{\partial}{\partial\phi^j_{KL}}W(\phi)\psi^i_{IJ}\psi^j_{KL} + \text{h.c.}\right),
\end{aligned} \tag{6.10}
$$

where we use the superfield conventions in [98]. We are interested in small fluctuations about a reference D-brane state, so we expand $W(\phi)$ in a Taylor series in ϕ.

All of the terms of interest in eq. (6.10) will be of the form

$$\mathrm{tr}\left[w_{i_1 i_2 i_3 \dots i_n}\left(F^{i_1}\phi^{i_2}\dots\phi^{i_n} - \frac{1}{2}\psi^{i_1}\psi^{i_2}\phi^{i_3}\dots\phi^{i_n}\right)\right].$$

The coefficients w may also depend on the closed-string background. We will examine small fluctuations about some reference background so that we can sensibly expand w in a Taylor series in fluctuations of the closed-string background.

The worldvolume fermions are represented in the open string theory by dimension 1 boundary Ramond vertex operators constructed from spin fields. In the $(-1/2)$ picture they can be written as:

$$V^{(-1/2)}_{R,IJ} = \zeta^{\alpha}_{i,a} e^{-\phi/2} S_{\alpha} \Sigma^{i} t^{a}_{IJ}\,, \tag{6.11}$$

where S_α is the spacetime part of the spin field and has dimension $1/4$; Σ^i is the internal part of the boundary vertex operator and has dimension $3/8$; ζ carries polarization and gauge indices; and t^a_{IJ} are the Chan-Paton matrices. Since we are interested in computing a potential term we are interested in zero-momentum amplitudes, so we can omit spacetime momentum factors $e^{ik\cdot X}$. Note that the spacetime directions will have standard Dirichlet or Neumann conditions, so that S_α is easily related to its bulk counterpart, for example by the doubling trick.[14]

Similarly, the worldvolume scalars are represented by NS vertex operators. In the (-1) picture they can be written as

$$V^{(-1)}_{NS,IJ} = \zeta_{i,a} e^{-\phi} \psi^{i} t^{a}_{IJ}\,, \tag{6.12}$$

where $e^{-\phi}$ has dimension $1/2$, and ψ is a dimension $1/2$ boundary operator arising from the internal sector. To find the 0-picture operator, we find the superpartner of ψ under the (gauged) worldsheet $\mathcal{N}=1$ SUSY,

$$T_F(z)\psi^i(w) = \frac{1/2}{(z-w)}\mathcal{O}^i(w)\,, \tag{6.13}$$

where $T_F = \frac{1}{\sqrt{2}}(G^+ + G^-)$ is the gauged $\mathcal{N}=1$ part of the $\mathcal{N}=2$ superconformal currents. Here $\mathcal{O}^i$ has dimension 1. The vertex operator is simply $\mathcal{O}^i$; if it is exactly marginal its integral over the boundary is a valid conformal deformation of the worldsheet action. Note that if ψ^i is a chiral primary,

$$G^+(z)\psi^i(w) = (\text{non}-\text{singular terms})$$

as $z \to w$, and may write

$$G^-(z)\psi^i(w) = \frac{1/2}{(z-w)}\mathcal{O}^i\,.$$

[14] c.f. [99] for a nice discussion of this method.

The internal part of the vertex operators for the auxiliary fields, in the (0) picture, can be constructed from the internal part of the (-1)-picture scalar vertex operators, via the spectral flow operator mapping the NS sector back to itself [100]:

$$V_{\mathrm{aux},IJ} = \lim_{z\to w}\left[(z-w)e^{-i\sqrt{3}H}V^{(-1)}_{NS,IJ}\right]. \tag{6.14}$$

Essentially this is because one gets the auxiliary component by acting on the scalar fields twice with the spacetime SUSY current.

For deformations preserving spacetime SUSY, the internal part of the vertex operators should be constructed from the chiral ring of the $\mathcal{N}=2$ algebra.[15] This is because the marginal (0)-picture operators will have vanishing R-charge; thus they may be added to the worldsheet lagrangian while maintaining $\mathcal{N}=2$ worldsheet supersymmetry. The (-1)-picture operators will have charge $q=1$; the $(-1/2)$-picture Ramond operators will have charge $q=-1/2$; and the auxiliary fields will have charge $q=-2$.

We will also want to include bulk vertex operators in order to measure the effects of the closed-string background. For SUSY-preserving deformations, we will be interested in marginal operators in the (c,c) or (a,c) ring. The vertex operators for massless fields can be constructed from dimension $(1/2,1/2)$ operators $\psi(z,\bar z)$ in the internal sector, with charge $(1,1)$ if they are in the (c,c) ring, $(-1,1)$ if they are in the (a,c) ring, and so on. In the (-1) picture these operators are:

$$V^{(-1)}_{\mathrm{bulk}} = e^{-\phi(z)-\phi(\bar z)}\psi(z,\bar z)\,. \tag{6.15}$$

The $(0,0)$ picture operators can be constructed via the $\mathcal{N}=2$ supercurrents which cancel the U(1) charge, i.e.

$$V^{(0,0)}_{\mathrm{bulk}}(w,\bar w) = \oint dz \oint d\bar z G^-(z)G^-(\bar z)\psi(w,\bar w) \tag{6.16}$$

for (c,c) operators, and

$$V^{(0,0)}_{\mathrm{bulk}}(w,\bar w) = \oint dz \oint d\bar z G^+(z)G^-(\bar z)\psi(w,\bar w) \tag{6.17}$$

for (a,c) operators.

Now we wish to calculate the tree-level contribution to the nth order term of the superpotential; we will expand out the coefficients to kth order in the closed string fields. We are particularly interested in the case $n>2$, as we are studying putative moduli. We will examine the contribution of this term to the fermion bilinear part of the action. On the disc, we must fix 3 real moduli due to the $\mathrm{SL}(2,\mathbb{R})$ symmetry. In addition, we must absorb the superconformal ghost number violation on the disc.

[15] Or antichiral ring. We will fix the overall sign ambiguity of the U(1) charges by demanding that the boundary operators be chiral.

These requirements can be met by using the $(-1/2)$ picture for the two fermionic vertex operators and placing them at opposite sides of the disc, or equivalently at $z = 0$ and $z = \infty$ in the upper half plane. Furthermore we will take one of the NS vertex operators to be in the (-1) picture and fix its location between the two R vertex operators, i.e. at $z = 1$ in the upper half-plane. The remaining open- and closed-string vetex operators are in the (0) and $(0, 0)$ pictures, and are integrated respectively over the boundary and bulk of the worldsheet. The resulting amplitude on the disc is:

$$\begin{aligned}
A = \lim_{\delta_i,\epsilon_i\to 0}\Big\langle & e^{-\phi/2} S_\alpha \Sigma^{i_1}_{I_1 I_2}(x_3) \int_{x_2+\epsilon_2}^{x_3-\delta_2} dy_1 \mathcal{O}^{(i_2,(0))}_{I_2 I_3}(y_1) \int_{x_2+\epsilon_3}^{y_2-\delta_3} dy_2 \mathcal{O}^{(i_3,(0))}_{I_3 I_4}(y_2)\ldots \times \\
& \times e^{-\phi}\psi^{i_k(-1)}_{I_k I_{k+1}}(x_2)\ldots e^{-\phi/2} S_\beta \Sigma^{i_\ell}_{I_\ell I_{\ell+1}}(x_1)\ldots \int_{x_3+\epsilon_{n-3}}^{y_{n-4}-\delta_{n-3}} dy_{n-3}\mathcal{O}^{(i_n,(0))}(y_{n-3}) \times \\
& \times \int_D dz_1 \ldots dz_k \mathcal{O}^{1(0,0)}(z_1,\bar{z}_1)\ldots \mathcal{O}^{k(0,0)}(z_k,\bar{z}_k)\Big\rangle
\end{aligned} \tag{6.18}$$

where $\epsilon_2 > \epsilon_3 > \cdots > \epsilon_{k-1},\ \epsilon_{k+2} > \cdots > \epsilon_{\ell-1},\ \epsilon_{\ell+1} > \cdots > \epsilon_{n-3}$; this prescription of the limits of integration ammounts to a point-splitting regularization on the boundary. In addition we sum over all orderings consistent with the Chan-Paton indices; amplitudes with adjacent operators $\phi_{IJ}\phi_{KL}$ are only nonvanishing if $K = L$. For gauge-invariant amplitudes we would sum over all such indices and thus over all orderings.

So far we have not specified which of the four chiral rings the closed-string vertex operators live in. We will discuss below how operators in the different rings may or may not couple to these amplitudes for a given boundary condition.

We will also be interested in superpotential terms which are linear in the superfields and contain couplings to closed-string moduli, such as the last term in eq. (6.2). This term will not show up as a fermion bilinear; only the auxiliary field will in fact couple. Such a term can be computed on the disc with a single closed-string insertion and a single open-string insertion. $\mathrm{SL}(2,\mathbb{R})$ invariance allows us to fix the positions of both vertex operators. In addition, if we place the closed-string vertex operator in the $(-1,-1)$ picture and the open-string operator in the (0)-picture we have absorbed the superconformal ghost number violation (the left- and right-moving ghost zero modes will be tied together by the boundary condition.) The relevant tree-level amplitude is thus:

$$\langle e^{-\phi-\bar{\phi}}\psi^{(-1,-1)}(z,\bar{z}) V_{F,II}(x)\rangle. \tag{6.19}$$

All of these prescriptions allow us to perform tree-level calculations for fixed boundary conditions in the Gepner models. In the rest of this section we will discuss these amplitudes in general compactifications as correlators in the topologically twisted version of the internal CFTs. In this language we can revisit our question regarding Kähler decoupling from the superpotential of B-type branes.

6.3 Topological CFT with boundaries

We begin by reviewing and generalizing the discussions in refs. [101, 102, 103] of topological CFTs with boundary.

Topological CFTs can be constructed from $\mathcal{N}=2$ CFTs via "twisting" the stress tensor with the U(1) current [104]; that is, we define a new stress tensor:

$$T^{\rm top}(z) = T(z) \pm \frac{1}{2}\partial J(z)\,. \qquad (6.20)$$

Note the sign ambiguity; as we will discuss, the overall sign is physically unimportant but the relative sign between left- and right-moving sectors is physically meaningful. This twisting may be achieved by adding a charge of $\pm c/3$ at infinity; the change in the stress tensor is simply the shift derived in the Feigin-Fuchs construction. In closed string theories one can see this most simply by adding to the action the coupling of the U(1) current to a background gauge field $A = \frac{1}{2}\omega$ where ω is the worldsheet spin connection [103]. In the cases we are interested, where $J = i\sqrt{3}\partial H$, the term

$$\int d^2z \frac{1}{2}\left(J\bar{\omega} + \bar{J}\omega\right) \qquad (6.21)$$

can be integrated by parts to get a coupling of H to the Riemann curvature. For amplitudes on the sphere, one may use conformal invariance to write the sphere as a flat cylinder with two hemispherical caps. The initial and final states are created by the path integral on those caps with any operator insertions one might have there; the curvature on these hemispherical caps means that the above terms in the lagrangian become the half-unit spectral flow operators applied to the initial and final states.

If one constructs the open-string case via the doubling trick on the Riemann sphere, one finds again that the topological twisting is equivalent to an amplitude with half-units of spectral flow applied to the initial and final states. More generally, to derive the twisted theory on a surface with boundary via the above coupling to the background field, one must take the boundary contribution in $\int J\bar{\omega}$ + c.c. into account. If we rewrite the disc as a long strip with two caps, the background charge will be concentrated on the boundary of these caps and the result will again be spectral flow applied to the in- and out-states [103]. Care should be taken with any boundary operator insertions on or near this part of the boundary, as they may have contact terms with the charge insertion.

The relative sign of the twisting of the holomorphic and anti-holomorphic parts of the stress tensor comes from the relative sign of the background charge. The "A-model" arises from an axial twisting while "B-model" arises from a vector twisting [105]. In the presence of D-branes, the twisting must be compatible with the boundary conditions. We can see easily that A-type boundary conditions are compatible with the A-model and B-type boundary conditions are compatible with the

B-model, as in each case the "twisted" stress tensor satisfies

$$T^{\mathrm{top}} = \bar{T}^{\mathrm{top}}$$

and thus satisfies sensible boundary conditions.

In these twisted models, the conformal dimensions of $\mathcal{N} = 2$ primary operators are shifted by half their U(1) charge, with the sign depending on the twisting. In the B-model, G^+ and $\bar{G}^+$ become dimension-zero "scalar" Grassman operators and suitably define BRST currents. The NS (c, c) operators are annihilated by them and have dimension 0 with respect to T^{top}. These operators are denoted the "topological" operators and their correlators are independent of position, as one can see using the conformal Ward identities in the presence of the background charge. We denote them as the "0-form" operators $\mathcal{O}^{i(0)}$. In the closed string theory, we can also define "$(1,0)$-" and "$(0,1)$-form" operators

$$\begin{aligned}
\oint_{z\to w} dz G^-(z)\mathcal{O}^{(0)}(w,\bar{w}) &= \mathcal{O}^{(1,0)}(w,\bar{w}) \\
\oint_{\bar{z}\to\bar{w}} d\bar{z}\bar{G}^-(\bar{z})\mathcal{O}^{(0)}(w,\bar{w}) &= \mathcal{O}^{(0,1)}(w,\bar{w}) ,
\end{aligned} \tag{6.22}$$

and "2-form" (or "$(1,1)$-form") operators:

$$\oint_{z\to w}\oint_{\bar{z}\to\bar{w}} dz d\bar{z} G^-(z)\bar{G}^-(\bar{z})\mathcal{O}^{(0)}(w,\bar{w}). \tag{6.23}$$

We can see that the (0)-form operators are simply the internal parts of the $(-1,-1)$ operators of the untwisted theory, while the $(1,1)$-form operators are the internal parts of the $(0,0)$-picture operators of the untwisted theory. The operators

$$\oint dz \mathcal{O}^{(1,0)} ; \qquad \oint d\bar{z}\mathcal{O}^{(0,1)} ; \qquad \int d^2 z \mathcal{O}^{(1,1)},$$

are BRST-invariant on Riemann surfaces without boundary. On surfaces with boundary, the integrated 2-form operator is only BRST-invariant up to an integral of the one-form operators along the boundary, as one can see by integrating by parts; similarly the BRST transformations of the one-form operators pick up boundary terms if the curve of integration ends on a boundary of D.

One may similarly construct topological operators on the boundary from the chiral primary boundary operators. From these one may also construct "one-form" operators from the commutators or anticommutators of the operator with modes of the spin-2 operators G:

$$\frac{1}{\sqrt{2}}\left\{ G^-_{-1/2} + \bar{G}^-_{-1/2} , \mathcal{O}^{(0)} \right\} \tag{6.24}$$

In the cases that we can construct the boundary condition and boundary operators via the doubling trick, these can be written as closed-string one-form operators via holomorphic contour integrals, as above. Again, the integral of these operators along the boundary are BRST-invariant, up to potential contact terms with other boundary operators.

One may similarly construct BRST-invariant operators in the A-model with A-type boundary conditions. If the CFTs correspond to geometric Calabi-Yau sigma-models, then we can see following refs. [105, 6] that the open-string A-model describes D-branes wrapped around special lagrangian submanifolds and the topological closed-string operators are the Kähler deformations of the target space; while the B-model describes D-branes wrapped around holomorphic cycles, and the topological closed-string operators correspond to complex structure deformations of the target space. Note that although refs. [101, 103] discuss only the case of purely Neumann boundary conditions for the B-model, our general discussion shows that we may couple this topological theory to any supersymmetric, even-dimensional brane. (Mirror symmetry requires this, if we are allowed to discuss any supersymmetric 3-cycle in the mirror). Note also that in this geometric picture, the almost-BRST-invariance of the integrated 2-form observables makes sense: a change of complex structure (Kähler class) will change the definition of holomorphic (special lagrangian) submanifolds.

Now let us return to the fermion bilinear part of the $(n > 2)$th order superpotential. By stretching the cylinder out into the capped strip (figure 1) we may write the amplitude as the expectation value of some set of NS vertex operators between Ramond states; these states are created by applying the Ramond vertex operators to the vacuum. $\mathcal{N} = 2$ worldsheet supersymmetry allows us to write the internal-CFT part of these states as the spectral flow operator applied to NS operators acting on the vacuum;

$$\Sigma^i(0)|\mathrm{vac}\rangle = e^{-i\sqrt{3}H/2}\psi^i(0)|\mathrm{vac}\rangle. \tag{6.25}$$

The amplitude (6.18) factorizes into three pieces. The first is the superghost piece; the second is the two-point function of the spin fields polarized in the spacetime directions. These give essentially universal answers which we can expect from 4d Lorentz invariance. The internal CFT amplitude is the interesting part. It is an expectation value of n chiral or antichiral NS boundary operators and k bulk NS-NS operators in one of the four closed-string chiral rings, with two additional half-unit spectral flow operators each mapping NS states to R states. The fixed boundary operators become 0-form observables and the integrated boundary operators, 1-form observables. If the closed-string operators are (c, c) for the B-type twisting, corresponding to complex structure deformations, they become (almost)-invariant topological observables. If they are (a, c) operators, corresponding to Kähler deformations, they are exact with respect to the left-moving BRST current and one might hope that they decouple. We will address this issue below.

The result is (up to the caveats above) a correlator of topological operators in the topologically twisted theory. The fixed operators become 0-form observables and

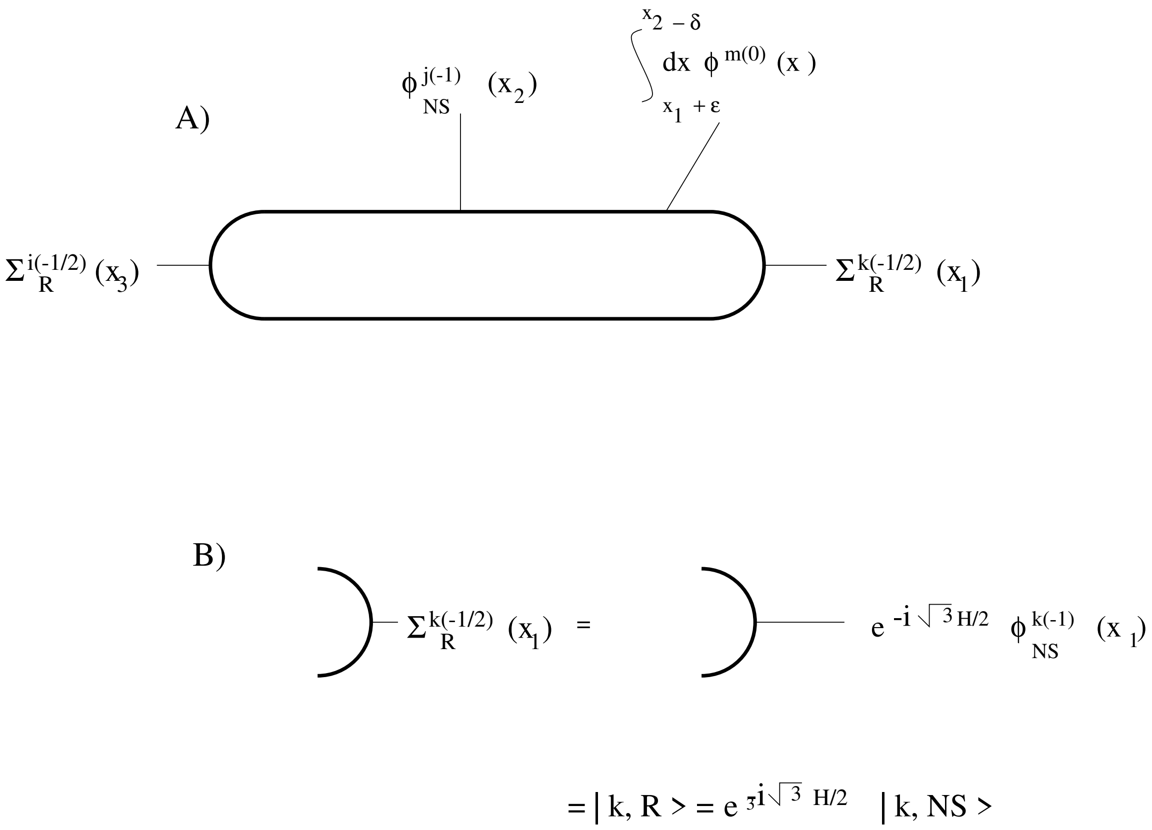

Figure 1: Computing the disc contribution to the D-brane superpotential. **A)**. CFT contribution. R vertex operators on "caps" can be written as R ground states. **B)**. Representation of R ground state as path integral on half-disc. Spectral flow maps this to NS ground state created by insertion of NS vertex operator.

the integrated operators become 1-form and 2-form observables. We may bring the techniques of topological field theory to bear on this calculation, and will do so below.

Similarly, the computation of eq. (6.19) is topological (with the same caveats). Here the auxiliary field is related by a full unit of spectral flow to the (0)-form observable of the associated scalar field. The superghost part of the amplitude merely takes care of the relevant zero modes. The internal CFT part is once again an amplitude in the topologically twisted theory of a (0)-form boundary observable and a $(0,0)$-form bulk observable.

6.4 Computations in the geometric sigma model

The topological symmetry of these correlators, and the localization properties of the topological path integrals [106, 105], make the above calculations relatively straightforward. To see this we will compute the cubic part of the superpotential for a D0-brane in a weakly curved background, and discuss the linear part of the superpotential for generic wrapped B-branes.

To begin with we need to construct the relevant topological observables. The closed-string case has been described in ref. [105] and the open-string case for fully Neumann boundary conditions has been described in [101]. We need to generalize these results to arbitrary B-model boundary conditions.

In the untwisted sigma-model, the propagating worldsheet fields are 3 complex scalars ϕ^i, and three complex fermions $\psi^i_\pm$. ψ^i has U(1) charge 1 and $\psi^{\bar{\imath}}$ has charge -1. Thus in the B-twisted theory ψ^i have dimension zero and become worldsheet scalars, while $\psi^{\bar{\imath}}$ have dimension one and become worldsheet one-forms. The BRST currents G^+, $\bar{G}^+$ give rise to global symmetries parameterized by constant Grassman scalars ϵ,$\bar{\epsilon}$ (since these are scalars such constants are well-defined on any worldsheet). In order to write these transformations in the simplest form, it is convenient to rewrite the fermions as:

$$\xi^i = \psi^i_+ + \psi^i_- \theta_{\bar{\jmath}} = g_{i\bar{\jmath}} \left(\psi^i_- - \psi^i_+\right) . \tag{6.26}$$

If we integrate out the auxiliary fields on the worldsheet, the BRST transformations become

$$\begin{aligned} \delta_{\rm B}\phi^i &= \frac{i}{2}\epsilon \left(\xi^i + g^{i\bar{\jmath}}\theta_{\bar{\jmath}}\right) \\ \delta_{\rm B}\phi^{\bar{\imath}} &= 0 \\ \delta_{\rm B}\xi^i &= i\Gamma^i_{jk}\left(\epsilon\psi^i_+\psi^k_- + \bar{\epsilon}\psi^j_-\psi^k_+\right) \\ \delta_{\rm B}\theta_{\bar{\jmath}} &= ig_{i\bar{\jmath}}\Gamma^i_{jk}\left(\epsilon\psi^j_+\psi^k_- - \bar{\epsilon}\psi^j_-\psi^k_+\right) + g_{i\bar{\jmath},k}g^{i\bar{\ell}}(i\epsilon\psi^i_+ + i\bar{\epsilon}\psi^i_-)\theta_{\bar{\ell}} \\ \delta_{\rm B}\psi^{\bar{\jmath}}_- &= -\bar{\epsilon}\bar{\partial}\phi^{\bar{\jmath}}\delta_{\rm B}\psi^{\bar{\jmath}}_+ = -\epsilon\partial\phi^{\bar{\jmath}} . \end{aligned} \tag{6.27}$$

These do not necessarily close off-shell once we have integrated out the auxiliary fields. In the presence of a boundary we must set $\epsilon = \bar{\epsilon}$.[16] Then the transformations simplify: in particular, the important transformations are:

$$\begin{aligned} \delta_{\rm B}\phi^i &= i\epsilon\xi^i \\ \delta_{\rm B}\xi^i &= 0\delta_{\rm B}\theta_{\bar{\jmath}} = 0 . \end{aligned} \tag{6.28}$$

Recall that in the Dirichlet directions of the untwisted model, ψ^i is fixed and $\psi^i_+ = -\psi^i_-$; in the Neumann directions (when we have turned off the NS 2-form and boundary gauge field), $\psi_+ = \psi_-$. Thus along Dirichlet directions, ξ vanishes at the boundary; while along Neumann directions, θ vanishes at the boundary. Of course, for curved boundaries, whether a given polarization is "Dirichlet" or "Neumann" will depend on ϕ; this can be defined by a projection matrix $\mathbb{P}^i_j(\phi(\mathcal{C})) : TM \longrightarrow T\mathcal{C}$.

The (0)-form topological observables in the bulk were constructed in [105]. They are of the form:

$$\Lambda_{i_1\ldots i_p}{}^{\bar{\jmath}_1\cdots\bar{\jmath}_q}\xi^{i_1}\ldots\xi^{i_p}\theta_{\bar{\jmath}_1}\theta_{\bar{\jmath}_q} ; \tag{6.29}$$

Λ is a $\bar{\partial}$-closed $(0,p)$ form with values in $\wedge^q T^{(0,1)}M$.

[16] This is a sensible thing to do in the closed-string sector as well, as B-model path integrals localize onto constant maps into the target space [105].

The boundary observables will live on the appropriate holomorphic submanifold $\mathcal{C} \subset M$ and will tak e values in the Chan-Paton algebra $\hat{g}$. In the fully Neumann case, the boundary observables are $\hat{g}$-valued $(p, 0)$-forms, while in the fully Dirichlet case they will be antiholomorphic functions of the position of the boundary, with values in $\wedge^q T^{(0,1)} M \otimes \hat{g}$. In the 2-brane and 4-brane cases, the observables will be forms on $\mathcal{C}$ valued in the normal bundle times the gauge group of the internal worldvolume, $N\mathcal{C}|_M \otimes \hat{g}$. The 0-form boundary observables corresponding to the marginal, chiral primaries of the untwisted model are linear in the worldsheet fermions. For all of these observables, open and closed, it is easy to see that the BRST operator acts as the holomorphic differential on $\mathcal{C}$. Thus topological observables are ∂-closed, and trivial BRST-exact observables are ∂-exact.

The construction of these topological amplitudes makes it clear that we have some anomalous U(1) charge. This essentially counts fermion number in these models. In the closed-string B-model, nonvanishing correlators have fermion number 3 for both θ and ξ corresponding to zero modes for each of these fields. On the disc, the boundary conditions will kill the ξ zero modes polarized along the Dirichlet directions and the θ zero modes polarized along the Neumann directions. Thus for a holomorphic p-cycle, nonvanishing correlators will have ξ fermion number p and θ fermion number $3 - p$.

The other fact that makes these amplitudes straightforward to calculate is that the topological path integral localizes onto constant maps, restricted to the submanifold defined by the boundary conditions. The correlation functions are then integrals of the appropriate forms over the moduli space of constant maps; in these cases they will be integrals of the pullback of forms on M onto the submanifold $\mathcal{C}$.

Let us start by computing the cubic term in the superpotential for a brane sitting at a point in the CY. The topological boundary observables corresponding to the world-volume chiral fields will be

$$\mathcal{O} = \phi_a^{\bar{\imath}} \theta_{\bar{\imath}} t^a \,, \tag{6.30}$$

where t^a is a matrix in the adjoint of the Chan-Paton group. The correlation function is:

$$\phi_a^{\bar{\imath}_1} \phi_b^{\bar{\imath}_2} \phi_c^{\bar{\imath}_3} \left(\langle \theta_{\bar{\imath}_1} \theta_{\bar{\imath}_2} \theta_{\bar{\imath}_3} \rangle + \langle \theta_{\bar{\imath}_2} \theta_{\bar{\imath}_1} \theta_{\bar{\imath}_3} \rangle \right) \operatorname{tr}(t^a t^b t^c). \tag{6.31}$$

Note that the expectation value is antisymmetric in the fermions; thus this will vanish if there is only one D-brane, after summing over the ordering. The moduli space of constant maps is a point. Chiral deformations of the location by boundary observables live in $T^{0,1}M$ and anti-chiral deformations are valued in $T^{1,0}M$. The latter are BRST-exact in our picture, so the above correlator is ∂-closed as a function of the location of the point.[17] The correlators must be antiholomorphic functions of

[17] This is a sensible thing to do in the closed-string sector as well, as B-model path integrals localize onto constant maps into the target space [105].

the location of this point, and they live in $\wedge^3 T^{(0,1)}M$. Serre duality implies that they are components of a closed antiholomorphic $(0,3)$ form. On the Calabi-Yau manifold there is only one such form $\overline{\Omega}$ within cohomology. Thus the superpotential is

$$W = \overline{\Omega}_{\bar{\imath}_1\bar{\imath}_2\bar{\imath}_3} f^{abc} \Phi_a^{\bar{\imath}_1} \Phi_b^{\bar{\imath}_2} \Phi_c^{\bar{\imath}_3} \,. \tag{6.32}$$

This superpotential is similar to the term coming from fully Neumann boundary conditions. The topological string theory in this latter case has been argued to be a holomorphic six-dimensional version of Chern-Simons theory [101]; the vertex operators which describe chiral fields in the spacetime lagrangian describe antiholomorphic gauge fields $\bar{A}$ on the Calabi-Yau. The low-energy lagrangian has been argued to be

$$S = \int_M \Omega \wedge \left(\bar{A} \wedge \partial \bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A} \right). \tag{6.33}$$

This second term is the superpotential.

Finally, we can look for linear terms in the superpotential coming from a coupling to closed strings. Again, the boundary topological operator will be linear in the worldsheet scalar fermions; for the correlator to have the right fermion number, the closed-string operator must be quadratic. We can analyze these couplings for 0-, 2-, 4- and 6-cycles separately and we find that some of these amplitudes vanish automatically.

For boundaries living on points, the open-string operator is written in eq. (6.30). The closed-string operator must be quadratic in θ and have no ξ charge by U(1) charge conservation. This latter operator is an element of $H^0(M, \wedge^2 T^{(1,0)}M)$. By Serre duality this group is equivalent to the Dolbeaux cohomology group $H^{(0,1)}(M)$ which vanishes for Calabi-Yau compactifications. Thus there is no closed-string operator which couples to a single open-string operator on the D0-brane. The argument is almost identical for D6-branes and rests on the fact that $H^{(2,0)}(M)$ is trivial on a Calabi-Yau manifold.

For 2-cycles the story is a bit richer. If the open-string vertex operator is polarized along a Neumann direction,

$$V_N = A_{i,a} \xi^i t^a \,, \tag{6.34}$$

(we work in a coordinate patch where the tangent-normal split is trivial), then fermion number conservation requires that the closed-string operator be quadratic in θ and there are no such nontrivial operators as we have just argued. But for vertex operators polarized in a Dirichlet direction, the closed-string operator must be bilinear in ξ and θ, making it a one-form valued in $T^{(0,1)}M$. Serre duality relates this to an element of $H^{(2,1)}(M)$ and this group is certainly nontrivial, so these open-closed correlators are allowed. Similarly we can have nontrivial linear terms for chiral fields coming from Neumann directions along a 4-cycle.

Indeed, these results should not surprise us. If we change the complex structure of the manifold, the holomorphic 2- and 4-cycles, and the homomorphic bundles on them, will change. We should generically find that the reference cycle is no longer a stable, supersymmetric configuration. On the other hands, the 0- and 6-cycles are holomorphic regardless of the complex structure, so we expect them to be supersymmetric so long as the closed-string background maintains $\mathcal{N}=2$ spacetime SUSY.

6.5 Decoupling of non-topological moduli

One of the more powerful statements one can make in topological closed string theory is that the Kähler (complex structure) deformations decouple from topological amplitudes in the B (A) model. This is related to the fact that the spacetime of the theory has $\mathcal{N}=2$ supersymmetry and the vector multiplets and hypermultiplets decouple (away from singular points in the moduli space). One can show in the topologically twisted B (A) models that insertions of integrated (c,a) $((a,c))$ operators, which one would get by taking derivatives of the amplitudes with respect to the Kähler (complex structure) moduli, lead to vanishing amplitudes. In the open string case the status of this decoupling is less clear. To start with, the spacetime SUSY is only $\mathcal{N}=1$ and the lagrangian is far less constrained. For example, recall the analogous $E_8\times E_8$ heterotic string, compactified on a CY 3-fold. There the charged multiplets arising from the Kähler and complex structure decouple from each other at finite order in α' [44] but couple due to worldsheet instantons [107]. Furthermore, in the generic $(0,2)$ model it may not make sense to identify deformations with Kähler or complex structure deformations.

Actually, a total decoupling is not to be expected, even from geometric considerations. For example, if we consider the theory of D9-branes wrapped on the CY, the four-dimensional action will come with the prefactor V_6/g_s where V_6 is the volume of the CY. On the other hand this is a B-brane so the topological amplitudes naturally depend on complex moduli. Thus the strongest conjecture we could make is that the superpotential (for a B brane) takes the form

$$W=m(\phi_K)W(\phi_c,\psi), \tag{6.35}$$

where $m(\phi_K)$ is proportional to the brane tension (3.1).

A known example which illustrates this is the one-loop topological open string amplitude. For the D6-brane this is the Ray-Singer torsion $I(V)$ associated to the Chan-Paton bundle V on M [103]. In general $I(V)$ is not independent of the Kähler moduli, but ratios $\ln(I(V_1)/I(V_2))$ are, where $V_{1,2}$ are two different bundles on M [108]. This is consistent with (6.35); furthermore this amplitude also corresponds to a chiral ($\int d^2\theta$) term in the effective action, the one-loop correction to the gauge coupling.

In a system involving several different branes, (6.35) does not even predict a universal multiplicative dependence of the total superpotential on the Kähler moduli. At the very least it will be the sum of several terms of this form but with different $m(\phi_K)$. There will also be terms involving strings stretched between different branes. Geometrically these would be expected to come with $m(\phi_K)$ for the surface of intersection; it would be quite interesting to make a more general proposal along these lines.

In any case, it is preferable to have string world-sheet arguments for decoupling. Thus we proceed to consider the cubic and quartic terms in the superpotential as computed on the disk, to see if derivatives with respect to Kähler moduli ϕ_K are consistent with (6.35). We will work in the untwisted theory in order to ensure that we are not avoiding any subtleties; our statements can be carried over to the twisted theory.

Recall that the cubic term in the superpotential for B-type boundary conditions is calculated via the 3-point disc amplitude. The part arising from the internal CFT is:

$$\langle \Sigma_{IJ}^{i_1} \mathcal{O}_{JK}^{i_2} \Sigma_{KI}^{i_3} \rangle \tag{6.36}$$

plus a sum over any orderings consistent with the Chan-Paton factors. We may fix the ordering by picking suitable Chan-Paton factors, which we will do here. Now the first derivative of this amplitude with respect to some Kähler deformation will lead to the above amplitude with the insertion of an integrated $(0,0)$-picture vertex operator constructed from the (a,c) (or (c,a)) ring:

$$V = \int_D d^2w \oint_{z\to w} \oint_{\bar{z}\to\bar{w}} dz d\bar{z} G^+ \bar{G}^-(\bar{z}) \psi^i(w,\bar{w})\,, \tag{6.37}$$

and the complete amplitude is show in figure 2. Conformal invariance allows us to deform the integral of G^+ out to the boundary. This amounts to using the superconformal Ward identities. Let us concentrate on the case where the doubling trick allows us to describe amplitudes on the upper-half plane via amplitudes on the full complex plane. The contour may be deformed to a sum of integrals of G^+ around each boundary operator[18], plus a contour integral around the image of (see figure 2)

$$V_{(a,c)}^{(0,1)} = \int d^2w \oint d\bar{z} \bar{G}^-(\bar{z}) \psi^i(w,\bar{w}). \tag{6.38}$$

In the end, the contour integrals around the boundary operators will vanish as the operators are chiral. The contour integral around the image of the bulk operator in the lower half-plane may be expressed in the upper half-plane as an integral

$$\oint_{\bar{z}\to\bar{w}} \bar{G}^+(\bar{z})$$

[18] Taking some care with the branch cuts created by the spin fields.

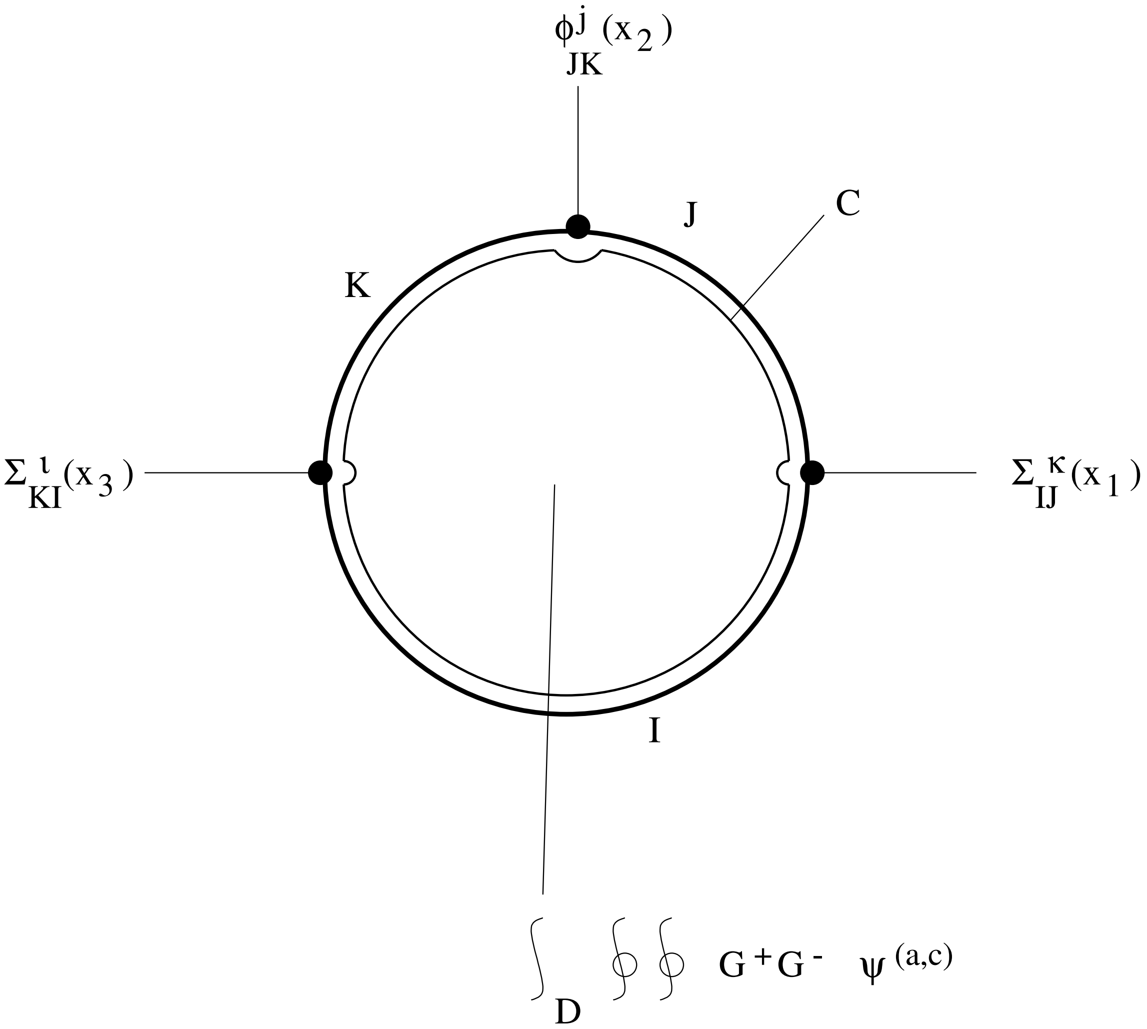

Figure 2: Perturbation of cubic part of superpotential by Kähler deformation. The superconformal Ward identities allow us to pull $\oint dzG^+(z)$ to the contour C. This can be deformed to a contour integral around each of the boundary operators and an integral over C of $d\bar{z}\bar{G}^+(\bar{z})$ which can be deformed back to the insertion of $\bar{G}^-\psi$. The result is an integral over the insertion of $\bar{\partial}_w\psi$ which can be integrated by parts to an integral of ψ over C.

around $V^{(0,1)}_{(a,c)}$. Using the superconformal algebra, this term becomes:

$$\int d^2w\bar{\partial}_w\psi^i = \oint_{\partial D}\psi^i\,. \tag{6.39}$$

In this integral over the boundary, we must take some care when the contour passes near one of the boundary operator insertions. The result is the correlation function

$$\left\langle \Sigma^{\alpha_1}_{IJ}\mathcal{O}^{\alpha_2}_{JK}\Sigma^{\alpha_3}_{KI}\oint_C\psi^i\right\rangle, \tag{6.40}$$

where the contour C is shown in figure 2. We get two potential contact terms from this correlator. One arises from the operator product of the (a, c) operator with the boundary [14]:

$$\lim_{y\to 0}\psi^i(x+iy, x-iy) \sim C^I_{\psi^i\mathcal{O}^\alpha}\frac{1}{y^{1-\delta_{\mathcal{O}^\alpha}}}\mathcal{O}^\alpha(x) \tag{6.41}$$

(here I lables the boundary condition in the region of the contact term); the other from the operator product of ψ^i with the boundary operators $\Sigma^\alpha, \mathcal{O}^\alpha$. Note that $\mathcal{O}$ will have zero U(1) charge. Let us deal with each of these in turn; we will in fact argue that this second contact term is taken care of by the first.

The bulk-boundary OPE can be treated as a factorization of the disc amplitude onto an intermediate open-string state. The OPE coefficient $C^I_{i\alpha}$ will be proportional to the open-closed disc amplitude $\langle\psi^i\mathcal{O}^\alpha\rangle$. There are in fact two classes of terms to worry about in eq. (6.41): $\delta_{\mathcal{O}^\alpha} < 1$ and $\delta_{\mathcal{O}^\alpha} = 1$. In the former case the intermediate state is a tachyon. Either this is removed by the GSO projection, or the perturbation by ψ^i has changed the acceptable boundary conditions — for example by changing the stability condition on vector bundles — so that the original boundary is no longer a stable D-brane. Such a divergence will have to be removed by perturbing the boundary conditions. The second case is a more genuine contact term; it is a dimension-one operator which is integrated over the boundary. In this case the 3-point correlator satisfies:

$$\left(\partial_i - C^{\mathrm{bound}}_{i\alpha}\partial_\alpha\right)\langle\Sigma^{\alpha_1}_{IJ}\mathcal{O}^{\alpha_2}_{JK}\Sigma^{\alpha_3}_{KI}\rangle = 0\,. \tag{6.42}$$

If $\mathcal{O}^\alpha$ is a topological operator then the perturbation by ψ^i has the fairly simple effect of moving the vev slightly along a flat direction. It should not affect the form of the superpotential, in keeping with our claim.

A very similar formula to (6.42) appears in [6]. In that case they find that by defining a suitable connection, the chiral primary part of the boundary state of the B-type brane is covariantly constant with respect to deformations of the Kahler moduli. Our result should be the open-string version of this fact.

The second contact term above is between the bulk operator ψ^i and boundary operator $\mathcal{O}^\beta$. By using the doubling trick this is described as the coalescence of three operators and, associativity allows us to write this by taking the bulk-boundary OPE (6.41) of ψ first, and then taking the OPE of $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$. But this will be included in the limits of integration of the first contact term $\mathcal{O}^\alpha$ over the boundary.

Higher order amplitudes are more subtle since they have a moduli space of insertions of vertex operators. When applying the Ward identities, we will find integrals of total derivatives with respect to these moduli, leading to contributions from the boundaries of moduli space.[19] Indeed such terms are important in deriving the one-loop holomorphic anomaly for topological amplitudes [110]. We can already illustrate

[19] This is similar to the fact that insertions of the stress tensor into correlators on higher-genus Riemann surfaces lead not only to transformations of the operators but to derivatives of the ampli-

this phenomenon by looking at the contribution to the quartic term of the superpotential from a single derivative with respect to the Kähler moduli. The resulting amplitude is:

$$\lim_{\epsilon,\delta\to 0}\Big\langle \Sigma^{i,(-1/2)}(x_3)\int_{x_2+\delta}^{x_3-\epsilon} dx\frac{1}{\sqrt{2}}\left\{G^- + \bar{G}^-, \mathcal{O}^{j,(-1)}(x)\right\}\times \tag{6.43}$$

$$\times\, \mathcal{O}^{k,(-1)}(x_2)\Sigma^{\ell,(-1/2)}(x_1)\int d^2w \oint_{z\to w}\oint_{\bar{z}\to\bar{w}} dz d\bar{z} G^+(z)\bar{G}^-(\bar{z})\psi^{(a,c)}\Big\rangle$$

plus a potential sum over orderings. As before we may fix the orderings of the boundary operators via a judicious choice of Chan-Paton factors. Once again, we pull the contour integral of G^+ off of the bulk operator and apply the superconformal Ward identities. In addition to the terms which we have already argued to vanish, we get a term coming from the contour integral of G^+ around the integrated NS operator. Again, let us look at the case where we may describe this amplitude via the doubling trick. Then the above anticommutator can be replaced with a contour integral of G^- around a point on the real line, and the contour integral of G^+ around this leads simply to a derivative of ψ^j. The result is the difference of contact terms:

$$\lim_{\epsilon\to 0}\Big\langle \Sigma^i(x_3)\left(\mathcal{O}^j(x_3-\epsilon) - \mathcal{O}^j(x_2+\epsilon)\right)\mathcal{O}^k(x_2)\Sigma^\ell(x_1)\times$$

$$\times\int d^2w\oint_{\bar{z}\to\bar{w}} d\bar{z}\bar{G}^-(\bar{z})\psi^{(a,c)}(w,\bar{w})\Big\rangle. \tag{6.44}$$

In the twisted theory we might hope that factorization and associativity means that this difference would vanish. This would be true if there was no insertion of $\psi^{(a,c)}$. With such an insertion, it is not clear that the amplitude will factorize onto topological intermediate states, so we cannot complete this argument at present.

The upshot of all of this is that there is a simple world-sheet mechanism which could lead to decoupling. It is very analogous to the known decoupling of bulk Kähler and complex structure deformations: the decoupling operator is a descendant with respect to an operator which annihilates the boundary chiral fields (say for Kähler and B-type, the operator G^+). The situation is better than that for $(0,2)$ heterotic string models as there are still two $\mathcal{N}=2$ algebras involved; they are identified only on the boundary.

Such world-sheet arguments are valid up to the possible contributions of contact terms and to make them precise, one needs to show that the contact terms either vanish or have simple interpretations (e.g. as connection coefficients on the moduli space). We have interpreted some but not all of these terms and thus can say that we have found further evidence for decoupling but by no means a proof.

tude with respect to the moduli of the surface [109]. Indeed such terms are important in deriving the one-loop holomorphic anomaly for topological amplitudes [110].

7. Correlation functions in minimal models and Gepner models

We now turn to the problem of computing correlation functions in the Gepner model. To begin with, let us recall a few properties of Gepner model boundary correlators, which are comparable to properties of bulk correlators. As with correlators in the bulk theory, in the boundary theory there are restrictions due to ghost number conservation. This can easily be seen using the doubling trick and has been discussed in the previous section. In addition, the boundary fields transform under particular representations of the chiral algebra, similar to chiral halves of bulk fields. The chiral algebra is the tensor product of the chiral algebras of the minimal models involved. The fields obey the same fusion rules. Correlators forbidden by the fusion rules therefore vanish.

In this section we point out a number of differences with bulk theory computations and interpret their consequences.

7.1 Ordering effects

Correlation functions involving boundary operators require a specification of operator ordering along the boundary (which we will place on the real line):

$$\langle\psi_1(x_1)\psi_2(x_2)\ldots\psi_n(x_n)\rangle \quad x_1 > x_2 > \cdots > x_n$$

This ordering corresponds directly to the ordering of the matrix fields in the world-volume lagrangian: for example terms $\operatorname{tr}\psi_1\psi_2\psi_3$ and $\operatorname{tr}\psi_1\psi_3\psi_2$ come from these two orderings of the three-point function.

In some particularly simple models (for example, free field theory), correlation functions of boundary operators can be analytically continued to the bulk. In this case it is possible to determine the effect of arbitrary permutations of the fields. This was formalized by Recknagel and Schomerus [111] in a discussion of non-supersymmetric conformal field theories. Two boundary operators $\psi_{1,2}$ were called mutually local if

$$\psi_1(x_1)\psi_2(x_2) = \psi_2(x_2)\psi_1(x_1) \tag{7.1}$$

inside correlators. Here, the left hand side implies $x_1 > x_2$ and the right hand side $x_1 < x_2$. Recknagel and Schomerus then argued that self-local marginal boundary operators are truly marginal. The argument is basically that an o.p.e.

$$\psi(x_1)\psi(x_2) \rightarrow \frac{1}{x_1 - x_2}\psi(x_1) + \cdots$$

of the form which would spoil marginality is incompatible with (7.1).

Free fermion correlators can be continued into the bulk as well, and in section 6 we saw that the superpotentials governing these operators were completely antisymmetric. In particular they vanish in the theory of a single brane. It seems quite plausible that this result applies to all operators which are strictly marginal in the large volume limit; however since we do not know whether an operator we find at the Gepner point is marginal in the large volume limit until we compute the superpotential (and many interesting operators are never strictly marginal), such considerations appear somewhat circular.

In general, one does not expect either that boundary correlation functions have a continuation into the bulk or that the boundary operators have such simple exchange relations. By general principles (which we review in the next subsection) boundary correlation functions in minimal models and Gepner models are particular combinations of several chiral conformal blocks, each of which has different exchange relations, chosen to be single-valued on the boundary. To make any statement about ordering effects, we must consider this analysis.

7.2 Sewing constraints

In this section we will briefly discuss sewing constraints on boundary fields. Correlation functions in two-dimensional CFT with boundaries have been studied for rational conformal field theories in [112, 14]. In the bulk, the n-point functions on the sphere are determined by the three-point functions; the higher-point functions can be computed by sewing. The result is independent of the decomposition of the n-point function into three-point functions, as guaranteed by crossing symmetry for the four-point functions. Similar results hold for the case with boundaries. Here, we have three types of sewing constraints, those involving only boundary fields, those, involving both bulk- and boundary fields and those involving only bulk fields. The structure constants for boundary fields depend on the boundary conditions.

As discussed in section 4, for RCFTs the possible boundary conditions preserving all the symmetries are labeled by the primary fields and can be implemented by boundary states carrying these labels, and we have written analogous states for Gepner models. The field content of the theory can be read off from the partition function $Z_{\alpha\beta}$; thus the propagating fields also carry the labels α, β. In the case that $\alpha \neq \beta$, the field $\phi^{\alpha\beta}$ is a boundary condition-changing operator. If $\alpha = \beta$, it preserves the boundary condition.

Let us concentrate on the correlation functions for boundary fields. The boundary OPEs are:

$$\phi_i^{\alpha\beta}(x)\phi_j^{\beta\gamma}(y) = \sum_k c_{ijk}^{\alpha\beta\gamma}\phi_k^{\alpha\gamma}(y)(x-y)^{h_k-h_i-h_j} + \cdots \qquad y < x\,. \tag{7.2}$$

The structure constants $c_{ijk}^{\alpha\beta\gamma}$, together with the vacuum amplitude, determine the three-point functions:

$$\langle\phi_i^{\alpha\beta}(x_i)\phi_j^{\beta\gamma}(x_j)\phi_k^{\alpha\gamma}(x_k)\rangle = c_{ijk^\dagger}^{\alpha\beta\gamma}c_{k^\dagger k\mathbf{1}}^{\alpha\gamma\alpha}\langle\mathbf{1}\rangle_\alpha(x_i-x_j)^{h_k-h_i-h_j}(x_j-x_k)^{-2h_k}\,. \tag{7.3}$$

We can also evaluate the correlator in the other channel:

$$\langle \phi_i^{\alpha\beta}(x_i)\phi_j^{\beta\gamma}(x_j)\phi_k^{\alpha\gamma}(x_k)\rangle = c_{ii^\dagger\mathbf{1}}^{\alpha\beta\alpha} c_{jki^\dagger}^{\beta\gamma\alpha}\langle\mathbf{1}\rangle_\alpha (x_j-x_k)^{h_i-h_j-h_k}(x_i-x_j)^{-2h_i}\,. \tag{7.4}$$

The dependence on the coordinates is dictated by conformal symmetry. As mentioned above, conformal symmetry does not relate three-point functions with different orderings. Comparison of (7.3), (7.4) leads to a consistency condition on the structure constants.

In addition to these conditions on the OPE coefficients, we must demand the crossing symmetry of the four-point functions. Nonvanishing correlation functions for boundary fields are of the form

$$\langle \phi_1^{\alpha\beta}\phi_2^{\beta\gamma}\phi_3^{\gamma\delta}\phi_4^{\delta\alpha}\rangle, \tag{7.5}$$

as illustrated in figure 3. In the case of rational symmetric models the factorization conditions can be made explicit using the conformal blocks. Note that for boundary correlators, the four point functions are linear in the conformal blocks.

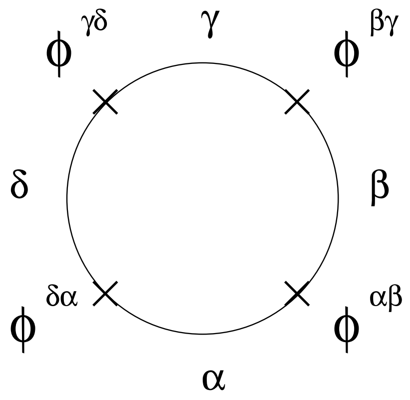

Figure 3: Four-point function.

In [113] an explicit solution was given for the Virasoro minimal models:[20]

$$c_{ijk}^{\alpha\beta\gamma} = F_{k\beta}\begin{bmatrix}\alpha & \gamma\\ i & j\end{bmatrix},$$

If we want to compare the four point functions for open string operators with different orderings, we have to take into account that the change in the ordering will in general require different boundary conditions for the respective four point function to be non-vanishing. Different boundary conditions will in general change both the structure constants and the expectation values of the identity so that we do not expect the four point functions to agree. In the example of the Virasoro minimal model, where we have an explicit solution, the boundary structure constants satisfy

$$c_{ijk}^{\alpha\beta\gamma} = c_{jik}^{\gamma\beta\alpha}. \tag{7.6}$$

In particular, they are completely symmetric in the case that there is only one boundary condition involved. The symmetry of the structure constant is of course a direct consequence of the symmetries of the F-matrices, which are specific for minimal models. (In general, there will be a phase involved [116].)

Another simple example is the U(1) boson. The primary fields are given by tachyon vertex operators e^{ikX}. The vertex operator e^{ikX} connects the boundary

[20](Note added in JHEP version): This was later proven in [114, 115].

conditions $|n\rangle\rangle$ to $|n+k\rangle\rangle$. Therefore, the condition (7.5) is fulfilled, whenever momentum conservation holds.

The sewing constraints determining the OPEs have not yet been solved for $\mathcal{N}=2$ minimal models or for Gepner models. We will return to this in future work.

7.3 Boundary selection rules

Given two boundary states, $|\alpha\rangle\rangle, |\beta\rangle\rangle$, the partition function will contain the characters of a particular set of marginal operators, whose insertion changes the boundary conditions from α to β. In general, this set of marginal operators will be a subset of all possible weight one representations, and are determined by the fusion rules [13]. As a consequence, certain correlators remain uncorrected, because the required marginal operator does not propagate with the particular boundary conditions. As a simple application, consider a boundary condition-changing marginal operator $\phi^{\alpha\beta}$. All non-vanishing correlators $\langle \phi_1^{\gamma\delta}\phi_2^{\delta\epsilon}\phi_3^{\epsilon\gamma}\rangle$ containing only boundary changing operators cannot be corrected by insertions of $\phi^{\alpha\beta}$. On the other hand, one can generate a non-zero correlator which vanishes at lower order.

7.4 Three-point functions in the Gepner models

A superpotential for massless fields is computed from $n>2$-point functions as we have discussed. Let us briefly discuss the conditions under which a three point function can be non-vanishing. To compute a $\langle\phi\phi F\rangle$ term we start by picking three vertex operators in the NS sector and we apply spectral flow by one unit to one of them. This is done by splitting the operators in a charged and an uncharged part as in (6.7) and applying the spectral flow $e^{-iH\sqrt{3}}$. This gives the following correlator:

$$\langle \mathcal{O}_{0_1}^{\alpha\beta} e^{-iH\frac{2}{\sqrt{3}}} \mathcal{O}_{0_2}^{\beta\gamma} e^{i\frac{H}{\sqrt{3}}} \mathcal{O}_{0_3}^{\gamma\delta} e^{i\frac{H}{\sqrt{3}}}\rangle \,. \tag{7.7}$$

Including the ghost contributions, the result is the product of the OPE coeffient $c_{123}^{\alpha\beta\gamma}$ for the uncharged operators, with the vacuum expectation amplitude for the boundary condition α. Thus, a cubic term in the superpotential is directly proportional to a structure constant $c_{123}^{\alpha\beta\gamma}$.

7.5 The A_1 model

The power of boundary selection rules can be nicely illustrated with the example of the A_1 model, where all correlators between chiral fields are forbidden. The model contains the chiral operators $\mathbf{1}$, $\phi_1^{(0)} = e^{\frac{i}{\sqrt{3}}\phi}$ and the antichiral operator $\phi_{-1}^{(0)} = e^{-\frac{i}{\sqrt{3}}\phi}$. As discussed in section 6, from these operators we can derive one-form operators. For the chiral operator, we get the one-form operator $\phi_1^{(1)} = e^{\frac{-2i}{\sqrt{3}}\phi}$. Alternatively, this operation can be interpreted as picture changing, if the A_1 model is part of a string theory compactification. A candidate for a non vanishing correlator is $\langle(\phi_1^{(0)})^3\phi_1^{(1)}\rangle$ and we have to check whether it is compatible with the boundary conditions. We can determine, which boundary conditions allow for the field ϕ_1 by using the fusion rules. The field $\phi_1^{\alpha\beta}$ exists whenever $N^1_{\alpha\beta}$ does not vanish. This is the case for $\alpha\beta = \mathbf{1}1, -1\mathbf{1}$ and $1-1$. As a consequence, our candidate is suppressed by boundary selection rules.

7.6 The A_2 model

In the A_2 model, the boundary selection rules do not forbid all correlators. We will give an example of an allowed correlator, where permutation of the operators requires different boundary conditions.

The A_2 minimal model can be seen as one real boson χ and one real fermion λ. The central charge is $3/2$. Apart from the identity there are two more chiral primary fields,

$$\phi_1 = \sigma e^{\frac{i\chi}{2\sqrt{2}}}, \quad \phi_2 = e^{\frac{i\chi}{\sqrt{2}}}.$$

There are two corresponding anti-chiral fields of opposite charges.

$$\phi_{-1} = \mu e^{-\frac{i\chi}{2\sqrt{2}}}, \quad \phi_{-2} = e^{\frac{-i\chi}{\sqrt{2}}}.$$

There is also an uncharged field λ, which is the ordinary fermion, or in minimal model language the field $l = 2, m = 0$. The spectrum for various boundary conditions can now be determined by fusing the fields labeling the boundary conditions.

The boundary conditions for the anti-chiral fields follow from this table: For any chiral field $\phi_q^{\alpha\beta}$ we have an antichiral field $\phi_{-q}^{\beta\alpha}$. A non-trivial four-point function to compute is

$$\langle \phi_2 \phi_2 \phi_1 \phi_1 \rangle,$$

where one operator is a one-form and integrated over the boundary. There exists another ordering

$\mathbf{1}$	$\mathbf{1}^{\mathbf{1},\mathbf{1}}$, $\mathbf{1}^{1,1}$, $\mathbf{1}^{-1,-1}$, $\mathbf{1}^{2,2}$, $\mathbf{1}^{-2,-2}\mathbf{1}^{\lambda,\lambda}$
ϕ_1	$\phi_1^{1,2}\phi_1^{-2,1}\phi_1^{1,-2}\phi_1^{2,-1}\phi_1^{-1,\lambda}\phi_1\lambda, 1$
ϕ_2	$\phi_2^{1,-1}\phi_2^{-1,1}\phi_2^{2,\lambda}\phi_2^{\lambda,-2}$
λ	$\lambda^{1,1}\lambda^{-2,2}\lambda^{2,-2}$

Table 2: Spectrum of boundary fields depending on the boundary conditions.

$$\langle \phi_2 \phi_1 \phi_2 \phi_1 \rangle.$$

The first ordering requires the following boundary conditions:

$$\langle \phi_2^{2,\lambda} \phi_2^{\lambda,-2} \phi_1^{-2,1} \phi_1^{1,2} \rangle$$

The other ordering requires the boundary conditions:

$$\langle \phi_2^{1,-1} \phi_1^{-1,\lambda} \phi_2^{\lambda,-2} \phi_1^{-2,1} \rangle.$$

Evaluation of the two four-point functions leads to structure constants with different boundary conditions (which will in general be not equal). In the final result, the expectation value of the identity is taken with two different boundary conditions. Therefore, the two results are not expected to agree.

7.7 The models $(k=2)^2$ and $(k=2)^4$, $(k=1)^3$ and $(k=1)^9$

There are two Gepner models containing only the $(k=2)$ model: the model $(k=2)^2$, which corresponds to a 2-torus; and $(k=2)^4$, which corresponds to a K3 surface. We will consider in this section the case of a single boundary condition (i.e.a single brane). In this case it is known that the superpotential vanishes, which we can check using the Gepner model description. The marginal operator in the $(k=2)^2$ model is the operator $\phi_2^{(1)}\phi_2^{(2)}$. This operator corresponds to the complex fermion ψ in sigma-model language. We know that this operator is an anticommuting variable. Therefore, all superpotential terms involving this operator vanish in the absence of Chan-Paton factors, due to the sum over operator ordering. Similarly, as a consequence of the fusion rules, in the $(k=2)^4$ case all marginal operators are of the form $\phi_2^{(i)}\phi_2^{(j)}$. We know from the torus that these operators anticommute. Therefore, all superpotential terms vanish after summing over all permutations. This verifies the result discussed in section 4.4. for the case $N_g=1$.

Similar statements can be made for the models consisting only of $(k=1)$ models, like the torus $(k=1)^3$ and the orbifolded six-torus $(k=1)^9$. The fermion in the two-torus is given by the field $\phi^{(1)}_{(1,1,0)}\phi^{(2)}_{(1,1,0)}\phi^{(3)}_{(1,1,0)}$. Simlilarly, we can form three complex fermions for the six-torus example. The marginal operators propagate for the $L=1$ boundary conditions in these models. Again, we know that the superpotential vanishes by antisymmetry.

7.8 The quintic

Let us now turn to the applications of the boundary selection rules to the $(k=3)^5$ Gepner model. We are particularly interested in correlators of marginal operators. For the B-boundary states discussed in section 5, we found that if we impose the same boundary conditions on both ends of the string there are boundary conditions with either 101, 50, 24, 11, 4 or 0 marginal operators propagating.

The 101 marginal operators propagating between the boundary states $L=\{1,1,1,1,1\}$ are of the same form as the complex structure deformations in the closed string case. These are left-right symmetric fields of charge $(1,1)$. If we use the doubling trick, the boundary marginal operators look the holomorphic part of these operators. They are of the form:

$(1,1,0)^5$	1
$(1,1,0)^3(2,2,0)(0,0,0)$	20
$(1,1,0)^2(3,3,0)(0,0,0)^2$	30
$(1,1,0)^2(2,2,0)^2(0,0,0)$	20
$(2,2,0)(3,3,0)(0,0,0)^3$	20

(7.8)

For these boundary conditions there are no further restrictions on possible correlators from the boundary selection rules. All correlators allowed by U(1) charge conservation and fusion rules will also be allowed by the boundary selection rules. The difference from closed string computations is that these correlators depend on: the one-point function of the identity in the presence of particular boundary conditions; the values of the fusion coefficients $c_{ijk}^{\alpha\alpha\alpha}$ for the boundary conditions; and the different integration domain for the four- and higher point functions.

The 4,11, 24 and 50 marginal operators are particular subsets of the 101 operators. Here, restrictions from boundary selection rules are possible: some of the correlators which are present in the closed string case cannot appear as the operators do not propagate with the given boundary conditions. We expect the strongest results for the case with 4 marginal boundary operators.

To compute superpotential terms, we start with three (or more) chiral primary fields in the NS sector. One way to relate this to a physical amplitude is to apply a unit of spectral flow, so that from the space time point of view we are computing the $\langle F\phi\phi\rangle$ term in the worldvolume lagrangian. For 4- and higher-point functions, charge conservation requires us to apply the operator G to the additional operators (which changes the label s of the Gepner model fields from 0 to 2). In our example, unit spectral flow is implemented by the operator $\phi^5_{(3,-3,0)}$.

The four marginal operators for the boundary conditions $L = \{1,0,0,0,0\}$ are given by

$$\psi_i = \phi^{(1)}_{(2,2,0)}\phi^{(i)}_{(3,3,0)}\,,$$

where the upper index labels the minimal model, $i = 2,3,4,5$. Spectral flow relates these operators to $\phi^{(1)}_{(1,-1,0)}\phi^{(i)}_{(0,0,0)}\phi^{3}_{(3,-3,0)}$. There is no non-vanishing three-point function for these operators. However, there are some higher-order terms which are allowed. The four point function

$$\langle\psi_2\psi_3\psi_4\int_{\partial\Sigma}G\psi_5\rangle$$

is not suppressed by the selection rules. We can ask about possible corrections to this correlator. The fusion rules tell us that all higher order terms have to be of the form

$$\left(\int_{\partial\Sigma}G\psi_j\right)^5\,,$$

since the fifth powers of $\phi_{(2,2,2)}$ and $\phi_{(3,3,2)}$ contain the identity in their fusion. Note that there are a lot more correction terms for the corresponding bulk 4-point function.

Let us move on to the next most complicated example, the example with 11 marginal operators. The marginal operators are of the form

$$\phi^{(1)}_{(2,2,0)}\phi^{(i)}_{(3,3,0)},\quad \phi^{(2)}_{(2,2,0)}\phi^{(i)}_{(3,3,0)},\quad \phi^{(1)}_{(1,1,0)}\phi^{(2)}_{(1,1,0)}\phi^{(i)}_{(3,3,0)}. \tag{7.9}$$

Compared to the previous case, this example is already much less restricted. For example, we have in this case two types of three-point functions: those containing each type of operator listed in (7.9) once; and that containing the three operators of the third type listed in (7.9). However, a lot of corrections which would be allowed in the corresponding bulk case are absent for these boundary conditions, because in the factors $3, 4, 5$ of the minimal models, $\phi_{(3,3,0)}$ is the only chiral primary which propagates.

Likewise, we potentially get more three point functions for the cases with 24 and 50 operators, and less corrections are suppressed. Finally, for the case with 101 marginal operators, all allowed bulk correlators have a boundary equivalent.

To conclude this section, let us briefly commment on the A-type boundary conditions. Here, the $L = 1$ states have one marginal operator, which is the operator $\phi^{(1)}_{(1,1,0)} \cdots \phi^{(5)}_{(1,1,0)}$. The three-point function between three operators of this type is allowed by the selection rules. Also, higher-order correlators containing $\left(\int G\left(\phi^{(1)}_{(1,1,0)} \cdots \phi^{(5)}_{(1,1,0)}\right)\right)^5$ are allowed. Taking the 5th power is required by the fusion rules: $\phi^5_{(1,1,0)}$ contains the identity. For the A-type boundary conditions we have also argued for a coupling of this operator to a bulk field. The closed string observables in the A-model are the (a, c) fields. On the quintic, this is the Kähler deformation $(\prod_{i=1}^5 \phi_{(1,1,0)}, \prod_{i=1}^5 \phi_{(1,-1,0)})$. Taking this operator to the boundary, we get boundary fields contained in the OPE of $\prod_{i=1}^5 \phi_{(1,1,0)} \times \prod_{i=1}^5 \phi_{(1,1,0)}$. This certainly makes a bulk-boundary coupling of the desired form possible.

7.9 Consequences of the selection rules

Perhaps the simplest conclusion we can draw from these selection rules is that the B branes with $L \geq 1$ have non-trivial moduli spaces. Consider the example of $|10000\rangle_B$: the superpotential must take the form

$$W = \psi_2\psi_3\psi_4\psi_5 f(\psi_2^5, \psi_3^5, \psi_4^5, \psi_5^5).$$

No matter what f is, the subspace $\psi_2 = \psi_3 = 0$ (or any two ψ's zero) solves $W' = 0$.

On the other hand, we found that the branes $|11111\rangle_A$, which we identified with the large volume $\mathbb{RP}^3$s, admitted a superpotential $W = \psi^3 f(\psi^5) + \phi\psi g(\psi^5)$, which would resolve the potential contradiction with the lack of moduli in the large volume limit. A non-trivial f and g would break the $\psi \to -\psi$ R symmetry of the leading order superpotential, which has no reason to exist in the large volume limit. However we cannot test the prediction for the number of minima of W at this point.

8. Conclusions and further directions

In this work we began a systematic study of D-branes in the stringy regime of the quintic Calabi-Yau. Our main result was the determination of the charges (in the usual large volume conventions) of the explicit Gepner model boundary states constructed by Recknagel and Schomerus. Our tools were the intersection form, and the monodromy and continuation formulas for the CY periods. The techniques clearly generalize to any Calabi-Yau given this data.

The primary question we hope to address is whether the spectra and low energy world-volume theories of branes in the stringy regime are the same (up to renormalizations of couplings) as in the large volume limit or not. We will refer to this as the "geometric hypothesis." Unlike the previously studied cases, supersymmetry is not sufficient to answer this question. From the bulk point of view, $\mathcal{N}=2$, $d=4$ supersymmetry allows lines of marginal stability for BPS states, while from the brane point of view $\mathcal{N}=1$, $d=4$ supersymmetry allows transitions from Higgs to Coulomb branch which are essentially unpredictable from the large volume point of view.

There are a number of considerations which would lead us to expect non-geometric phenomena. Perhaps the simplest is that the B monodromies relate branes of different dimension. Another essentially non-geometric phenomenon is the topology change seen in various Calabi-Yaus; both phenomena make the the geometric interpretation of brane probes ambiguous. The mere existence of these phenomena however does not really contradict the geometric hypothesis as we have stated it, if the different geometric objects related by monodromy and topology change lead to the same low energy theories. What we would be saying is that the same brane theory can have multiple large volume limits, a familiar phenomenon in duality.

Small instantons and the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold provide examples which do contradict the geometric hypothesis in its simplest form. At a special point (more generally, in complex codimension one) in moduli space, enhanced gauge symmetry and additional states appear. This might be considered a relatively mild failure as it is associated with a singularity of the riemannian geometry or gauge bundle. If all failures of the geometric hypothesis were associated with singularities, conversely it would be true under the mild condition that the geometry stayed non-singular. As we mentioned in the introduction, we can imagine much more drastic failures — a priori, the spectrum of branes at the Gepner point might have satisfied none of the relations we expected from geometry and gauge theory.

The results we have presented here are not (yet) inconsistent with the geometric hypothesis. Most of the branes we find could certainly correspond to the appropriate geometric constructions — holomorphic vector bundles and special lagrangian submanifolds. For example, all of the branes we found satisfied the (mathematical) stability condition on vector bundles. The lack of any classification of these makes it difficult for us to assert that branes which we did not identify actually do not have geometric constructions. The elliptically fibered case may be more promising in this regard. The most problematic case was a brane which would correspond to a rank 1 bundle with $c_1 = 0$ but $c_2 \neq 0$. Although such things do not exist in conventional gauge theory, they are known to exist in modified gauge theories (such as noncommutative gauge theory), so one can imagine that this object has a description in the large volume limit.

We also presented a general argument that B-type branes should be described by geometric considerations — namely, that their world-volume potentials are determined by quantities in B-twisted topological models, which are equal to their classical values in the B-twisted topological sigma models, up to hopefully minor effects of Kähler deformations. Besides a formal world-sheet argument we showed that many known cases fit with this idea. By contrast, the superpotentials in A brane theories can depend directly on Kähler moduli and a priori it would seem much more likely that the geometric hypothesis fails.

Finally, we made some first steps towards explicit computation of the superpotentials on these branes. These superpotentials appear to be quite non-trivial and it appears that such computations are doable with existing techniques; we will return to this in future work. An exciting possibility is that topological open string theory can be developed to the point where exact superpotentials can be obtained, perhaps with some analog of the special geometry determining the bulk prepotentials.

One important direction to develop is to find more direct ways to get the geometric interpretation of these states. The results here suggest that this will be simpler in the A picture — the simplest picture is that each component minimal model has a specific boundary condition for its LG superfield. If we had the D0-brane boundary state, we could apply a probe construction to get the geometrical picture for the B boundary states (indeed we could derive the corresponding $(0,2)$ models); perhaps larger classes of boundary states containing the D0 can be found.

A study of curves of marginal stability is in progress, to decide whether the large volume and Gepner D-branes should be expected to match up, and whether new phenomena appear near the conifold point.

Let us close with a brief discussion the physical relevance of our primary question. To the extent that branes in the stringy regime are qualitatively different than geometric branes, all of the work on compactification using branes will have to be reconsidered. On the other hand, to the extent that they are qualitatively the same, these techniques will provide new ways of deriving geometric results, such as the existence and moduli space dimension for vector bundles.

Questions of existence of branes are also directly relevant for non-perturbative constructions of M theory. For example, Matrix theory constructions to date use D0-branes as their starting point. Compactifications on some manifold M are believed to be described by D0-branes in a certain scaling limit of type-IIA string theory on M [117]. The authors of ref. [118] argued that for Calabi-Yau compactifications this limit was the mirror of the conifold point. If it were to turn out that the D0-brane did not exist in the stringy regime, this construction would have to be reconsidered.

In any case we believe there is much to be discovered in this direction.

JHEP08(2000)015

Acknowledgments

We would like to thank Paul Aspinwall, Tom Banks, Michael Bershadsky, Ralph Blumenhagen, Robert Bryant, Emanuel Diaconescu, Robbert Dijkgraaf, Dan Freed, Bartomeu Fiol, Jaume Gomis, Rajesh Gopakumar, Sheldon Katz, Albrecht Klemm, Maxim Kontsevich, Juan Maldacena, John Morgan, Hirosi Ooguri, Arvind Rajaraman, Andreas Recknagel, Moshe Rozali, Volker Schomerus, Shishir Sinha, Andy Strominger, Gang Tian, Cumrun Vafa, Edward Witten and Sasha Zamolodchikov for useful discussions and correspondence.

This research was supported by DOE grant DE-FG02-96ER40959. A.L. is also supported in part by NSF grant PHY-9802709 and DOE grant DE-FG02-91ER40654.

A. An explicit calculation for A-type boundary states

This appendix shows the explicit calculation of the intersection number of two A-type boundary states. The Witten index $\mathrm{tr}_R(-1)^F$ in the open string sector is obtained from the transition amplitude between the (internal) RR parts of the boundary states with $(-1)^{F_L}$ inserted[21]. The first part of this calculation is very close to that in [7].

For the A-type boundary states the δ_A constraint is trivial, as we have discussed.

$$\begin{aligned}\mathop{\mathrm{tr}}_R(-1)^F q^H &= \frac{1}{\kappa_\alpha^A \kappa_{\tilde\alpha}^A} \sum_{\lambda,\mu}^{\beta,R} \sum_{\tilde\lambda,\tilde\mu}^{\beta,R} B_{\tilde\alpha}^{\tilde\lambda,\tilde\mu} B_\alpha^{\lambda,\mu}\, {}_A\langle\langle \tilde\lambda, -\tilde\mu | (-1)^{F_L} \tilde q^{L_0-\frac{c}{24}} | \lambda,\mu\rangle\rangle_A \\ &= \frac{1}{\kappa_\alpha^A \kappa_{\tilde\alpha}^A} \sum_{\lambda,\mu}^{\beta,R} \sum_{\lambda',\mu'}^{ev} B_{\tilde\alpha}^{\lambda,-\mu} B_\alpha^{\lambda,\mu} (-1)^{Q(\mu)+d/2} S_{(\lambda,\mu),(\lambda',\mu')} \chi_{\mu'}^{\lambda'}(q), \qquad \text{(A.1)}\end{aligned}$$

where $S_{(\lambda,\mu),(\lambda',\mu')}$ is the modular transformation matrix and ev means $l_j + m_j + s_j =$ even. The β-constrained sum together with the charge projection operator can be rewritten as

$$\begin{aligned}\sum_{\lambda,\mu}^{\beta,R} (-1)^{Q(\mu)+d/2} &= \sum_{\lambda,\mu}^{R} \frac{1}{K} \sum_{\nu_0=0}^{K-1} e^{i\pi(2\nu_0+1)(-\sum \frac{s_j}{2} + \sum \frac{m_j}{k_j+2} + \frac{d}{2})} \\ &= \frac{1}{K} \sum_{\lambda,\mu}^{R} \sum_{\nu_0} e^{i\pi\frac{d}{2}(2\nu_0+1)} \prod_{j=1}^{r} e^{i\pi\frac{m_j}{k_j+2}(2\nu_0+1)} e^{-i\pi\frac{s_j}{2}(2\nu_0+1)}. \qquad \text{(A.2)}\end{aligned}$$

Putting all these equations together and collecting terms we get:

$$\mathop{\mathrm{tr}}_R(-1)^F q^H = \frac{1}{\kappa_\alpha^A \kappa_{\tilde\alpha}^A} \frac{1}{K} \sum_{\lambda',\mu'}^{ev} \sum_{\nu_0} e^{i\pi\frac{d}{2}(2\nu_0+1)} \prod_{j=1}^{r} \Big(\frac{1}{2(k_j+2)^2} \times$$

[21] One has to be careful with the definition of $(-1)^{F_L}$ in the RR sector. It should be defined by $(-1)^{F_L} = (-1)^{J_0+d/2}$ because the charge might be half- integer moded.

$$\times \sum_{l_j, m_j, s_j}^{R} \frac{\sin(l_j, L_j)\sin(l_j, \tilde{L}_j)\sin(l_j, l'_j)}{\sin(l_j, 0)} \times$$
$$\times e^{i\pi \frac{m_j}{k_j+2}(2\nu_0+1+M_j-\widetilde{M}_j+m'_j)} e^{i\pi\frac{s_j}{2}(-2\nu_0-1-S_j+\widetilde{S}_j-s'_j)}\Big)\chi^{\lambda'}_{\mu'}(q). \quad \text{(A.3)}$$

The sums over l_j, m_j, s_j can be evaluated as follows:

$$\frac{1}{2(k+2)^2}\sum_{l,m,s}^{R} \frac{\sin(l,L)\sin(l,\tilde{L})\sin(l,l')}{\sin(l,0)} e^{i\pi\frac{m}{k+2}M} e^{i\pi\frac{s}{2}S} =$$
$$= \frac{1}{(k+2)^2}\sum_{l,m}^{R} \frac{\sin(l,L)\sin(l,\tilde{L})\sin(l,l')}{\sin(l,0)} e^{i\pi\frac{m}{k+2}M}(-1)^{\frac{S}{2}}\delta^{(2)}_S$$
$$= \frac{1}{k+2}\sum_{l} \frac{\sin(l,L)\sin(l,\tilde{L})\sin(l,l')}{\sin(l,0)}(-1)^{\frac{M}{k+2}(l+1)}\delta^{(k+2)}_M(-1)^{\frac{S}{2}}\delta^{(2)}_S$$
$$= \frac{1}{2}N^{l'}_{L,\tilde{L}}\delta^{(2k+4)}_M\delta^{(2)}_S(-1)^{\frac{S}{2}}\,. \quad \text{(A.4)}$$

Inserting this result into (A.3) gives

$$\mathop{\mathrm{tr}}_R(-1)^F q^H = \frac{1}{\kappa^A_\alpha\kappa^A_{\tilde{\alpha}}}\frac{1}{K}\frac{1}{2^r}(-1)^{\frac{S-\tilde{S}}{2}}\sum_{\lambda',\mu'}^{R}\sum_{\nu_0}(-1)^{\frac{d}{2}(2\nu_0+1)} \times$$
$$\times \prod_{j=1}^{r} N^{l'_j}_{L_j,\tilde{L}_j}\delta^{(2k_j+4)}_{2\nu_0+1+M_j-\widetilde{M}_j+m'_j}(-1)^{\nu_0+\frac{1+s'_j}{2}}\chi^{\lambda'}_{\mu'}(q). \quad \text{(A.5)}$$

This fits with the normalization constant being $\kappa^A_\alpha = \sqrt{C/K2^r}$, where C is an extra integer constant depending on the specific model. It can be understood from the β constraints. Imposing the same spin structure on all subtheories reduces the number of states by a factor of $1/2^r$, the U(1) constraint gives another factor of $1/K$.

To simplify this result we have to use the fact that the R ground states are given by $\phi^l_{l+1,1}$ which are identified with $\phi^{k-l}_{-k+l-1,-1}$; only these states will contribute to the Witten index. We continue the upper index of fusion rule coefficients $N^l_{L,\tilde{L}}$ with a period of $2k+4$; we identify $N^{-l-2}_{L,\tilde{L}} = -N^l_{L,\tilde{L}}$; and we set $N^{-1}_{L,\tilde{L}} = N^{k+1}_{L,\tilde{L}} = 0$. This continuation is natural from the point of view of the Verlinde formula. Neglecting an overall factor of $(-1)^{\frac{d}{2}}$ we find that:

$$\mathop{\mathrm{tr}}_R(-1)^F q^H = \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}}\sum_{\nu_0}(-1)^{(d+r)\nu_0}\prod_{j=1}^{r}\sum_{m_j=0}^{2k_j+3} N^{-m'_j-1}_{L_j,\tilde{L}_j}\delta^{(2k_j+4)}_{2\nu_0+1+M_j-\widetilde{M}_j+m'_j}$$
$$= \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}}\sum_{\nu_0}(-1)^{(d+r)\nu_0}\prod_{j=1}^{r} N^{2\nu_0+M_j-\widetilde{M}_j}_{L_j,\tilde{L}_j}\,. \quad \text{(A.6)}$$

References

[1] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, *Mod. Phys. Lett.* **A 4** (1989) 2073;
J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017];
J. Polchinski, *Tasi lectures on D-branes*, hep-th/9611050.

[2] R.G. Leigh, *Dirac-Born-Infeld action from dirichlet sigma model*, *Mod. Phys. Lett.* **A 4** (1989) 2767.

[3] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[4] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[5] M.R. Douglas and G. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[6] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B 477** (1996) 407 [hep-th/9606112].

[7] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[8] M. Gutperle and Y. Satoh, *D-branes in Gepner models and supersymmetry*, *Nucl. Phys.* **B 543** (1999) 73 [hep-th/9808080].

[9] M. Gutperle and Y. Satoh, *D0-branes in Gepner models and $N = 2$ black holes*, *Nucl. Phys.* **B 555** (1999) 477 [hep-th/9902120].

[10] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) theory*, *Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[11] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, 2*, hep-th/9903031.

[12] P. Candelas, X.C. D.L. Ossa, P.S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21.

[13] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[14] J.L. Cardy and D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, *Phys. Lett.* **B 259** (1991) 274.

[15] E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, *κ-symmetry, supersymmetry and intersecting branes*, *Nucl. Phys.* **B 502** (1997) 149 [hep-th/9705040].

[16] E. Bergshoeff and P.K. Townsend, *Super D-branes*, *Nucl. Phys.* **B 490** (1997) 145 [hep-th/9611173].

JHEP08(2000)015

[17] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys.* **B 480** (1996) 265 [hep-th/9606139].

[18] V. Balasubramanian and R.G. Leigh, *D-branes, moduli and supersymmetry*, *Phys. Rev.* **D 55** (1997) 6415 [hep-th/9611165].

[19] J.A. Harvey and G. Moore, *On the algebras of BPS states*, *Comm. Math. Phys.* **197** (1998) 489 [hep-th/960901].

[20] K. Uhlenbeck and S.-T. Yau, *On the existence of hermitean Yang-Mills connections on stable vector bundles*, *Comm. Pure Appl. Math.* **39** (1986) 257; *A note on our previous paper: 'On the existence of hermitian Yang-Mills connections on stable vector bundles'*;
I. Brunner, *Comm. Pure Appl. Math.* **42** (1989) 703;
S.K. Donaldson, *Infinite determinants, stable bundles and curvature*, *Duke Math. J.* **54** (1987) 231.

[21] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer-Verlag, New York 1998.

[22] R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, *J. High Energy Phys.* **11** (1997) 002 [hep-th/9710230].

[23] E. Witten, *D-branes and K-theory*, *J. High Energy Phys.* **12** (1998) 019 [hep-th/9810188].

[24] M.B. Green, J.A. Harvey and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033].

[25] Y.-K.E. Cheung and Z. Yin, *Anomalies, branes and currents*, *Nucl. Phys.* **B 517** (1998) 69 [hep-th/9710206].

[26] D. Morrison, *The geometry underlying mirror symmetry*, in *Recent trends in algebraic geometry*, K. Hulek et al. eds., Cambridge University Press 1999, alg-geom/9608006.

[27] M.R. Douglas, *Branes within branes*, hep-th/9512077.

[28] F. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, *Math. USSR Izv.* **13** (1979) 499.

[29] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves,* Vieweg 1997.

[30] M. R. Douglas, work in progress.

[31] N.J. Hitchin, *The moduli space of special lagrangian submanifolds*, dg-ga/9711002.

[32] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B 479** (1996) 243 [hep-th/9606040].

[33] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, hep-th/9804131.

JHEP08(2000)015

[34] A. Karch, D. Lüst and A. Miemiec, $N = 1$ *supersymmetric gauge theories and supersymmetric 3-cycles*, *Nucl. Phys.* **B 553** (1999) 483 [hep-th/9810254].

[35] S. Donaldson and P. Kronheimer, *The geometry of four-manifolds,* Clarendon 1990.

[36] M.A. Luty and I.W. Taylor, *Varieties of vacua in classical supersymmetric gauge theories*, *Phys. Rev.* **D 53** (1996) 3399 [hep-th/9506098].

[37] D.R. Morrison, *Mirror symmetry and the type-II string*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 146 [hep-th/9512016].

[38] S. Katz, *On the finiteness of rational curves on quintic threefolds*, *Comp. Math.* **60** (1986) 151.

[39] A. Albano and S. Katz, *Lines on the Fermat quintic threefold and the infinitesimal generalized Hodge conjecture*, *Trans. Am. Math. Soc.* **324** (1991) 353.

[40] S. Katz, *Rational curves on Calabi-Yau threefolds*, in *Mirror symmetry I*, S.-T. Yau ed., American Mathematical Society and International Press 1999.
alg-geom/9312009.

[41] M. Reid, *Algebraic varieties and analytic varieties* in *Advanced studies in pure mathematics 1*, S. Iitaka ed., Kinokuniya 1983.

[42] T. Hübsch, *Calabi-Yau manifolds: a bestiary for physicists*, World Scientific, River Edge, 1992.

[43] J. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *J. High Energy Phys.* **12** (1997) 002 [hep-th/9711053].

[44] E. Witten, *New issues in manifolds of* SU(3) *holonomy*, *Nucl. Phys.* **B 268** (1986) 79.

[45] J. Distler and S. Kachru, $(0, 2)$ *Landau-Ginzburg theory*, *Nucl. Phys.* **B 413** (1994) 213 [hep-th/9309110].

[46] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198].

[47] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[48] R.P. Thomas, *An obstructed bundle on a Calabi-Yau threefold*, math.AG/9903034.

[49] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Birkhauser 1980.

[50] A. Ceresole, R. D'Auria, S. Ferrara and A.V. Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[51] P.S. Aspinwall, *The moduli space of $N = 2$ superconformal field theories*, 1994 Trieste lectures, published in *1994 Summer School in High Energy Physics and Cosmology*, E. Gava et al. eds., hep-th/9412115.

[52] B.R. Greene and M.R. Plesser, *Duality in Calabi-Yau moduli space*, *Nucl. Phys.* **B 338** (1990) 15.

[53] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[54] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Measuring small distances in $N = 2$ sigma models*, *Nucl. Phys.* **B 420** (1994) 184 [hep-th/9311042].

[55] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090];
B.R. Greene, D.R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, *Nucl. Phys.* **B 451** (1995) 109 [hep-th/9504145].

[56] J. Polchinski and A. Strominger, *New vacua for type-II string theory*, *Phys. Lett.* **B 388** (1996) 736 [hep-th/9510227].

[57] B.R. Greene and Y. Kanter, *Small volumes in compactified string theory*, *Nucl. Phys.* **B 497** (1997) 127 [hep-th/9612181].

[58] A. Sagnotti, *Surprises in open-string perturbation theory*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 332 [hep-th/9702093].

[59] J. Fuchs and C. Schweigert, *Branes: from free fields to general backgrounds*, *Nucl. Phys.* **B 530** (1998) 99 [hep-th/9712257].

[60] T. Banks, L.J. Dixon, D. Friedan and E. Martinec, *Phenomenology and conformal field theory or can string theory predict the weak mixing angle?*, *Nucl. Phys.* **B 299** (1988) 613.

[61] T. Banks and L.J. Dixon, *Constraints on string vacua with space-time supersymmetry*, *Nucl. Phys.* **B 307** (1988) 93.

[62] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and crosscaps to the superstring*, *Nucl. Phys.* **B 293** (1987) 83.

[63] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[64] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251.

[65] T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, *Mod. Phys. Lett.* **A 4** (1989) 161 erratum *ibid.* **4** (1989) 885.

JHEP08(2000)015

[66] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B 296** (1988) 757.

[67] D. Gepner, *Lectures on $N = 2$ string theory*, *Lectures at spring school on superstrings*, Trieste, Italy Apr 3-14, 1989.

[68] B.R. Greene, *String theory on Calabi-Yau manifolds*, `hep-th/9702155`.

[69] W. Boucher, D. Friedan and A. Kent, *Determinant formulae and unitarity for the $N = 2$ superconformal algebras in two-dimensions or exact results on string compactification*, *Phys. Lett.* **B 172** (1986) 316.

[70] A.B. Zamolodchikov and V.A. Fateev, *Disorder fields in two-dimensional conformal quantum field theory and $\mathcal{N} = 2$ extended supersymmetry*, *Sov. Phys. JETP* **63** (1986) 913.

[71] P.D. Vecchia, J.L. Petersen, M. Yu and H.B. Zheng, *Explicit construction of unitary representations of the $N = 2$ superconformal algebra*, *Phys. Lett.* **B 174** (1986) 280.

[72] S. Nam, *The Kac formula for the $N = 1$ and the $N = 2$ superconformal algebras*, *Phys. Lett.* **B 172** (1986) 323.

[73] V.A. Fateev and A.B. Zamolodchikov, *Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $\mathbb{Z}_n$ invariant statistical systems*, *Sov. Phys. JETP* **62** (1985) 215.

[74] Z. Qiu, *Nonlocal current algebra and $N = 2$ superconformal field theory in two-dimensions*, *Phys. Lett.* **B 188** (1987) 207.

[75] D.A. Kastor, E.J. Martinec and S.H. Shenker, *RG flow in $N = 1$ discrete series*, *Nucl. Phys.* **B 316** (1989) 590.

[76] E.J. Martinec, *Algebraic geometry and effective lagrangians*, *Phys. Lett.* **B 217** (1989) 431.

[77] C. Vafa and N. Warner, *Catastrophes and the classification of conformal theories*, *Phys. Lett.* **B 218** (1989) 51.

[78] E.J. Martinec, *Criticality, catastrophes and compactifications*, in *Physics and mathematics of strings*, L. Brink, D. Friedan and A.M. Polyakov eds., World Scientific, River Edge 1990.

[79] W. Lerche, C. Vafa and N.P. Warner, *Chiral rings in $N = 2$ superconformal theories*, *Nucl. Phys.* **B 324** (1989) 427.

[80] F. Ravanini and S.-K. Yang, *Modular invariance in $N = 2$ superconformal field theories*, *Phys. Lett.* **B 195** (1987) 202.

[81] Z. Qiu, *Modular invariant partition functions for $N = 2$ superconformal field theories*, *Phys. Lett.* **B 198** (1987) 497.

[82] G. Mussardo, G. Sotkov and M. STashnikov, $\mathcal{N}=2$ *superconformal minimal models*, *Int. J. Mod. Phys.* **A 4** (1989) 1135.

[83] A.B. Zamolodchikov and V.A. Fateev, *Operator algebra and correlation functions in the two-dimensional Wess-Zumino* $\mathrm{SU}(2)\times\mathrm{SU}(2)$ *chiral model*, *Sov. J. Nucl. Phys.* **43** (1986) 657.

[84] S. Mukai, *Symplectic structure of the moduli of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984) 101; see also the review article *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*, *Sugaku expositions* **1** (1988) 139.

[85] N. Seiberg, *Observations on the moduli space of superconformal field theories*, *Nucl. Phys.* **B 303** (1988) 286.

[86] T. Eguchi and A. Taormina, *Unitary representations of* $N=4$ *superconformal algebra*, *Phys. Lett.* **B 196** (1987) 75; *Character formulas for the* $N=4$ *superconformal algebra*, *Phys. Lett.* **B 200** (1988) 315; *On the unitary representations of* $N=2$ *and* $N=4$ *superconformal algebras*, *Phys. Lett.* **B 210** (1988) 125.

[87] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* $\mathrm{SU}(N)$ *holonomy*, *Nucl. Phys.* **B 315** (1989) 193.

[88] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* R^4 *and* $(2,0)$ *superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[89] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[90] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[91] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of* $(2,0)$ *superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [hep-th/9712117].

[92] M.R. Douglas, A. Kato and H. Ooguri, *D-brane actions on Kähler manifolds*, *Adv. Theor. Math. Phys.* **1** (1998) 237 [hep-th/9708012].

[93] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[94] S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183];
A. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four dimensions*, *Nucl. Phys.* **B 533** (1998) 199 [hep-th/9803015].

[95] J. Polchinski, *Tensors from K3 orientifolds*, *Phys. Rev.* **D 55** (1997) 6423 [hep-th/9606165];
M.R. Douglas, *Enhanced gauge symmetry in M(atrix) theory*, *J. High Energy Phys.*

07 (1997) 004 [hep-th/9612126];
D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013 [hep-th/9712230].

[96] C.G. Callan Jr. and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147].

[97] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [hep-th/9706109].

[98] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press 1992.

[99] J. Polchinski, *String theory* vols. I-II, Cambridge University Press 1998.

[100] J.J. Atick, L.J. Dixon and A. Sen, *String calculation of Fayet-Iliopoulos D-terms in arbitrary supersymmetric compactifications*, *Nucl. Phys.* **B 292** (1987) 109.

[101] E. Witten, *Chern-Simons gauge theory as a string theory*, in *The Floer memorial volume*, H. Hofer et al. eds., Birkhauser, Boston 1995.

[102] P. Hořava, *Equivariant topological sigma models*, *Nucl. Phys.* **B 418** (1994) 571 [hep-th/9309124].

[103] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-spencer theory of gravity and exact results for quantum string amplitudes*, *Comm. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[104] T. Eguchi and S.-K. Yang, $N = 2$ *superconformal models as topological field theories*, *Mod. Phys. Lett.* **A 5** (1990) 1693.

[105] E. Witten, *Mirror manifolds and topological field theory*, in *Mirror symmetry, 1*, S.-T. Yau ed., American Mathematical Society, 1998.

[106] E. Witten, *Topological sigma models*, *Comm. Math. Phys.* **118** (1988) 411.

[107] M. Dine, N. Seiberg, X.G. Wen and E. Witten, *Nonperturbative effects on the string world sheet*, *Nucl. Phys.* **B 278** (1986) 769; *Nonperturbative effects on the string world sheet, 2*, *Nucl. Phys.* **B 289** (1987) 319.

[108] D. Ray and I.M. Singer, *Analytic torsion and the laplacian on complex manifolds*, *Ann. Math.* **98** (1973) 154.

[109] T. Eguchi and H. Ooguri, *Conformal and current algebras on general Riemann surface*, *Nucl. Phys.* **B 282** (1987) 308.

[110] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[111] A. Recknagel and V. Schomerus, *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys.* **B 545** (1999) 233 [hep-th/9811237].

[112] D.C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, *Nucl. Phys.* **B 372** (1992) 654.

[113] I. Runkel, *Boundary structure constants for the a-series Virasoro minimal models*, *Nucl. Phys.* **B 549** (1999) 563 [hep-th/9811178].

[114] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036].

[115] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030];
The geometry of WZW branes, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[116] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Comm. Math. Phys.* **123** (1989) 177.

[117] A. Sen, *D0-branes on T_n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220];
N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[118] S. Kachru, A. Lawrence and E. Silverstein, *On the matrix description of Calabi-Yau compactifications*, *Phys. Rev. Lett.* **80** (1998) 2996 [hep-th/9712223].